PDEs, Homework #5
Problems: 1, 2, 4, 5, 8
due Wednesday, March 24

1. Consider the function $u\colon \mathbb{R} \to \mathbb{R}$ defined by

$$u(x) = \left\{ \begin{array}{cc} 0 & \text{if } x < 0 \\ \sin x & \text{if } x \geq 0 \end{array} \right\}.$$

Show that $u'' + u = \delta$ in the sense of distributions. Hint: you need to show that

$$\int_{-\infty}^{\infty} u(x)\varphi''(x)\,dx + \int_{-\infty}^{\infty} u(x)\varphi(x)\,dx = \varphi(0)$$

for all test functions φ; simplify the first integral and then integrate by parts.

2. Consider the heat equation $u_t = k\Delta u$ over a bounded region $A \subset \mathbb{R}^n$ subject to zero Dirichlet boundary conditions. Show that each solution has a decreasing L^2-norm:

$$\frac{d}{dt}\int_A u(x,t)^2\,dx \leq 0$$

at all times. Hint: differentiate, use the PDE and then use Green's identities.

3. Let $u(x,t)$ be a compactly supported solution of the wave equation $u_{tt} = \Delta u$. Show that its energy $E(t)$ is conserved, where

$$E(t) = \int_{\mathbb{R}^n} u_t(x,t)^2 + |\nabla u(x,t)|^2\,dx.$$

4. Consider the initial value problem for the Burgers' equation

$$u_t + uu_x = 0, \qquad u(x,0) = f(x) \tag{BE}$$

when $f(x) = \sin x$. Show that u is bounded at all times, whereas u_x is not.

5. Solve the initial value problem for the Burgers' equation (BE) when $f(x) = -1/x$.

6. Solve the initial value problem for the Burgers' equation (BE) when f is the function defined by $f(x) = 1$ if $x < 0$ and $f(x) = 0$ if $x > 0$.

7. Consider the initial value problem

$$u_t + g(u)u_x = 0, \qquad u(x,0) = f(x).$$

Show that some characteristic curves will intersect, unless $g(f(x))$ is increasing.

8. Let $a > 0$ be fixed. Solve the initial value problem (BE) in the case that

$$f(x) = \left\{ \begin{array}{cc} a & \text{if } x \leq 0 \\ a(1-x) & \text{if } 0 < x < 1 \\ 0 & \text{if } x \geq 1 \end{array} \right\}.$$