PDEs, Homework #4
Problems: 1, 2, 4, 7, 10
due Wednesday, March 3

1. Show that the average value of a harmonic function over a ball is equal to its value at the centre. In other words, show that a harmonic function u satisfies

$$u(x) = \frac{1}{\alpha_n r^n} \int_{|y-x|\leq r} u(y)\, dy$$

for all $x \in \mathbb{R}^n$ and all $r > 0$, where α_n is the volume of the unit ball in $\mathbb{R}^n$. Hint: write the integral in polar coordinates and use the mean value property for spheres.

2. Suppose that $A \subset \mathbb{R}^n$ is a bounded region and that $u\colon \mathbb{R}^n \to \mathbb{R}$ satisfies

$$\Delta u(x) = \lambda u(x) \quad \text{when } x \in A, \qquad u(x) = 0 \quad \text{when } x \in \partial A.$$

Show that u must be identically zero, if $\lambda \geq 0$. Hint: write down Green's identity for the integral of $u\Delta u$ and then simplify.

3. Show that $H' = \delta$ in the sense of distributions, where H is the Heaviside function

$$H(x) = \left\{ \begin{array}{ll} 0 & \text{if } x < 0 \\ 1 & \text{if } x \geq 0 \end{array} \right\}.$$

4. We say that $u\colon \mathbb{R}^n \to \mathbb{R}$ is subharmonic, if $\Delta u(x) \geq 0$ for all $x \in \mathbb{R}^n$. Show that u^2 is subharmonic whenever u is harmonic.

5. Find all harmonic functions $u(x, y)$ which have the form $u(x, y) = F(x/y)$.

6. Find the unique solution $u(x, y)$ of the Dirichlet problem

$$u_{xx} + u_{yy} = 0 \quad \text{when } x^2 + y^2 < a^2, \qquad u(x, y) = g(x, y) \quad \text{when } x^2 + y^2 = a^2.$$

7. Solve $u_{xx} + u_{yy} = 1$ in the annulus $a^2 \leq x^2 + y^2 \leq b^2$ subject to zero Dirichlet boundary conditions. Hint: looking for radial solutions, one ends up with an ODE.

8. Given a bounded region $A \subset \mathbb{R}^n$, show that the Dirichlet problem

$$\Delta u(x) = f(x) \quad \text{when } x \in A, \qquad u(x) = g(x) \quad \text{when } x \in \partial A$$

has at most one solution. Give one proof using the maximum principle and one using Green's first identity. Hint: if u, v are solutions, then $w = u - v$ is harmonic, so its min/max values are attained on ∂A; use Green's identity for the integral of $w\Delta w$.

9. Suppose u is harmonic in the unit disc $x^2 + y^2 \leq 1$ and such that $u(x, y) = x^2$ on the boundary $x^2 + y^2 = 1$. Determine the value of u at the origin.

10. Find the unique bounded solution of the Laplace equation $u_{xx} + u_{yy} = 0$ in the upper half plane $y \geq 0$ subject to the boundary condition $u(x, 0) = \operatorname{sign} x$.