PDEs, Homework #3
Problems: 1, 3, 4, 5, 9
due Wednesday, Feb. 10

1. Use Hölder's inequality to show that the solution of the heat equation

$$u_t = k u_{xx}, \qquad u(x,0) = \varphi(x) \tag{HE}$$

goes to zero as $t \to \infty$, if φ is continuous and bounded with $\varphi \in L^p$ for some $p \geq 1$. Hint: you will need to compute the L^q norm of the heat kernel for some $q \geq 1$.

2. Solve the Dirichlet problem for the heat equation on the half line. In other words, find the solution to (HE) when $x > 0$ and the boundary condition $u(0,t) = 0$ is imposed for all $t \geq 0$. Hint: argue as for the Neumann problem but use an odd extension.

3. Use a substitution of the form $v(x,t) = e^{at}u(x,t)$ to solve the initial value problem

$$u_t - k u_{xx} = bu, \qquad u(x,0) = \varphi(x)$$

when b is a constant. Hint: if a is chosen suitably, then v satisfies the heat equation.

4. Show that the heat equation (HE) preserves positivity: if the initial temperature $\varphi(x)$ is non-negative for all x, then the temperature $u(x,t)$ is non-negative for all x, t. Show that the same is true for the associated Dirichlet problem on the interval $[0, L]$. Hint: the first part is easy; use the maximum principle to prove the second part.

5. Solve the initial value problem (HE) in the case $\varphi(x) = e^{ax}$ for some $a \in \mathbb{R}$. Hint: the solution is given by a messy integral; if you complete the square and use a suitable substitution, then you will end up with the standard integral involving e^{-z^2}.

6. Prove uniqueness of solutions for the Neumann problem

$$u_t = k u_{xx}, \qquad u(x,0) = \varphi(x), \qquad u_x(0,t) = 0 = u_x(L,t). \tag{NP}$$

Hint: if u, v are both solutions, then $E(t) = \int_0^L [u(x,t) - v(x,t)]^2 \, dx$ is decreasing.

7. Show that the solution to (HE) is even in x, if the initial datum $\varphi(x)$ is even.

8. Show that the non-homogeneous heat equation on the real line

$$u_t - k u_{xx} = f(x,t), \qquad u(x,0) = \varphi(x)$$

has a unique solution which is given by Duhamel's formula

$$u(x,t) = \int_{-\infty}^{\infty} S(x-y,t)\varphi(y)\, dy + \int_0^t \int_{-\infty}^{\infty} S(x-y,t-\tau) f(y,\tau)\, dy\, d\tau.$$

9. Show that the average temperature $T(t) = \int_0^L u(x,t)\, dx$ is conserved for each solution of the Neumann problem (NP). State the most general boundary conditions for which the average temperature is conserved.