PDEs, Homework #2
Problems: 2, 7, 8, 9, 10
due Wednesday, Dec. 2

1. Show that the solution $u(x,t)$ of the initial value problem
$$u_{tt} = c^2 u_{xx}, \qquad u(x,0) = \varphi(x), \qquad u_t(x,0) = \psi(x) \tag{WE}$$
is even in x, if the initial data φ, ψ are even. Hint: show $u(-x,t)$ is also a solution.

2. Solve the Neumann problem for the wave equation on the half line. That is, find the solution to (WE) when $x > 0$ and the boundary condition $u_x(0,t) = 0$ is imposed for all $t \geq 0$. Hint: argue as for the Dirichlet problem but use an even extension.

3. Find all solutions $u = u(x,y)$ of the second-order equation $u_{xx} + 4u_{xy} + 3u_{yy} = 0$.

4. Show that the solution to the wave equation (WE) need not remain bounded at all times, even though it is initially bounded. Hint: take the initial data to be constant.

5. Solve the wave equation (WE) in the case that $\varphi(x) = x^2$ and $\psi(x) = x + 1$.

6. Suppose that $a < b$ and consider the Cauchy problem (WE) in the case that
$$\varphi(x) = \psi(x) = \left\{ \begin{array}{cc} 1 & \text{if } a \leq x \leq b \\ 0 & \text{otherwise} \end{array} \right\}.$$
Compute the limit $\lim_{t\to\infty} u(x,t)$ for each fixed $x \in \mathbb{R}$.

7. Solve the following non-homogeneous wave equation on the real line:
$$u_{tt} - c^2 u_{xx} = t, \qquad u(x,0) = x^2, \qquad u_t(x,0) = 1.$$

8. Use the substitution $v(x,t) = e^{\lambda t} u(x,t)$ to solve the initial value problem
$$u_{tt} - u_{xx} + 2\lambda u_t + \lambda^2 u = 0, \qquad u(x,0) = \varphi(x), \qquad u_t(x,0) = \psi(x)$$
on the real line. Hint: you should find that v satisfies the wave equation $v_{tt} = v_{xx}$.

9. Consider the wave equation with damping $u_{tt} - c^2 u_{xx} + d u_t = 0$ on the real line. Show that the energy is decreasing for all classical solutions of compact support, if $d > 0$.

10. Solve the Cauchy problem (WE) on the half line $x > 0$ when $\varphi(x) = \psi(x) = 1$ and the Dirichlet condition $u(0,t) = 0$ is imposed for all $t \geq 0$. Is your solution a classical one? Hint: there are different formulas for the cases $x > ct$ and $x \leq ct$.

11. Find the eigenfunctions and eigenvalues of $-\partial_x^2$ subject to Neumann boundary conditions on $[0, L]$. That is, find all nonzero functions $F(x)$ and all $\lambda \in \mathbb{R}$ such that
$$-F''(x) = \lambda F(x), \qquad F'(0) = F'(L) = 0.$$

12. Solve the wave equation $u_{tt} = 4u_{xx}$ on the interval $[0, \pi]$ subject to the conditions
$$u(x,0) = \cos x, \qquad u_t(x,0) = 1, \qquad u(0,t) = 0 = u(\pi,t).$$