Method of characteristics: a special case

Consider a first-order linear homogeneous PDE of the form

$$a(x, y)u_x + b(x, y)u_y = 0.$$ 

In order to solve it, one tries to find parametric curves

$$x = x(s), \quad y = y(s)$$ 

along which $$u(x, y)$$ remains constant. Such curves are given by the system of ODEs

$$\frac{dx}{ds} = a(x, y), \quad \frac{dy}{ds} = b(x, y), \quad \frac{du}{ds} = 0 \quad (3.1)$$

and they are usually called characteristic curves. Imposing the initial conditions

$$x(0) = x_0, \quad y(0) = y_0, \quad u(0) = u_0, \quad (3.2)$$

one can then solve the system (3.1)-(3.2) to determine the value of $$u(x, y)$$ at any point that lies on a characteristic curve through $$(x_0, y_0)$$.

Example 1. We use the method of characteristics to solve the problem

$$2u_x - u_y = 0, \quad u(x, 0) = f(x).$$

In this case, the characteristic equations are given by

$$\frac{dx}{ds} = 2, \quad \frac{dy}{ds} = -1, \quad \frac{du}{ds} = 0$$

so we can easily solve them to get

$$x = 2s + x_0, \quad y = -s + y_0, \quad u = u_0.$$ 

Imposing the initial condition $$u(x_0, 0) = f(x_0)$$, we now eliminate $$x_0$$ and $$s$$ to find that

$$y = -s \implies u = f(x_0) = f(x - 2s) = f(x + 2y).$$

Example 2. We use the method of characteristics to solve the problem

$$u_t + xu_x = 0, \quad u(x, 0) = g(x).$$

In this case, the characteristic equations

$$\frac{dt}{ds} = 1, \quad \frac{dx}{ds} = x, \quad \frac{du}{ds} = 0$$

imply that $$t = s + t_0, \ x = x_0e^s$$ and $$u = u_0$$. Since $$u(x_0, 0) = g(x_0)$$ by assumption, we get

$$t = s \implies u = g(x_0) = g(xe^{-s}) = g(xe^{-t}).$$
Method of characteristics: the general case

Consider a partial differential equation of the form

\[ a(x, y, u)u_x + b(x, y, u)u_y = c(x, y, u) \]  \hspace{1cm} (3.3)

and suppose that the values of \( u \) are known on some initial curve

\[ x(0) = x_0, \quad y(0) = y_0, \quad u(0) = u_0, \]  \hspace{1cm} (3.4)

where \( y_0, u_0 \) are given in terms of \( x_0 \). By solving the characteristic equations

\[ \frac{dx}{ds} = a, \quad \frac{dy}{ds} = b, \quad \frac{du}{ds} = c \]  \hspace{1cm} (3.5)

subject to the initial conditions (3.4), one can then obtain a unique local solution for the original system (3.3)-(3.4), provided that the Jacobian

\[ J = \det \begin{bmatrix} \frac{dx}{ds} & \frac{dy}{ds} \\ \frac{dx}{dx_0} & \frac{dy}{dx_0} \end{bmatrix} \]

is nonzero along the initial curve (3.4). In fact, the solvability condition \( J \neq 0 \) is equivalent to the condition that the initial curve (3.4) is not itself a characteristic curve.

**Example 3.** We use the method of characteristics to solve the problem

\[ xu_x + yu_y = 2u, \quad u(x, 1) = f(x). \]

In this case, the characteristic equations are

\[ \frac{dx}{ds} = x, \quad \frac{dy}{ds} = y, \quad \frac{du}{ds} = 2u \]

so we can easily solve them to get

\[ x = x_0e^s, \quad y = y_0e^s, \quad u = u_0e^{2s}. \]

In view of our initial condition \( u(x_0, 1) = f(x_0) \), this gives

\[ y = e^s \implies u = u_0e^{2s} = f(x_0)y^2 = f(x/y)y^2. \]

**Example 4.** We use the method of characteristics to show that the problem

\[ u_x + u_y = 1, \quad u(x, x) = 1 \]

has no solutions. In this case, the characteristic equations are \( x' = y' = u' = 1 \) and so

\[ x = s + x_0, \quad y = s + y_0, \quad u = s + u_0. \]

Since the line \( y = x \) is one of the characteristic curves, it is better to avoid it and impose some other initial condition. Using a generic one such as \( u(x_0, 0) = f(x_0) \), we find

\[ y = s = x - x_0 \implies u = s + f(x_0) = y + f(x - y), \]

so all solutions of the PDE must have this form. In particular, \( u(x, x) = x + f(0) \) cannot be equal to a constant and the given initial condition is never satisfied.