Method of characteristics: a special case

► Consider a first-order linear homogeneous PDE of the form

$$a(x,y)u_x + b(x,y)u_y = 0.$$

In order to solve it, one tries to find parametric curves

$$x = x(s), \qquad y = y(s)$$

along which $u(x,y)$ remains constant. Such curves are given by the system of ODEs

$$\frac{dx}{ds} = a(x,y), \qquad \frac{dy}{ds} = b(x,y), \qquad \frac{du}{ds} = 0 \tag{3.1}$$

and they are usually called characteristic curves. Imposing the initial conditions

$$x(0) = x_0, \qquad y(0) = y_0, \qquad u(0) = u_0, \tag{3.2}$$

one can then solve the system (3.1)-(3.2) to determine the value of $u(x,y)$ at any point that lies on a characteristic curve through (x_0, y_0).

Example 1. We use the method of characteristics to solve the problem

$$2u_x - u_y = 0, \qquad u(x,0) = f(x).$$

In this case, the characteristic equations are given by

$$\frac{dx}{ds} = 2, \qquad \frac{dy}{ds} = -1, \qquad \frac{du}{ds} = 0$$

so we can easily solve them to get

$$x = 2s + x_0, \qquad y = -s + y_0, \qquad u = u_0.$$

Imposing the initial condition $u(x_0, 0) = f(x_0)$, we now eliminate x_0 and s to find that

$$y = -s \quad \Longrightarrow \quad u = f(x_0) = f(x - 2s) = f(x + 2y).$$

Example 2. We use the method of characteristics to solve the problem

$$u_t + xu_x = 0, \qquad u(x,0) = g(x).$$

In this case, the characteristic equations

$$\frac{dt}{ds} = 1, \qquad \frac{dx}{ds} = x, \qquad \frac{du}{ds} = 0$$

imply that $t = s + t_0$, $x = x_0e^s$ and $u = u_0$. Since $u(x_0, 0) = g(x_0)$ by assumption, we get

$$t = s \quad \Longrightarrow \quad u = g(x_0) = g(xe^{-s}) = g(xe^{-t}).$$

Method of characteristics: the general case

▶ Consider a partial differential equation of the form

$$a(x, y, u)u_x + b(x, y, u)u_y = c(x, y, u) \tag{3.3}$$

and suppose that the values of u are known on some initial curve

$$x(0) = x_0, \qquad y(0) = y_0, \qquad u(0) = u_0, \tag{3.4}$$

where y_0, u_0 are given in terms of x_0. By solving the characteristic equations

$$\frac{dx}{ds} = a, \qquad \frac{dy}{ds} = b, \qquad \frac{du}{ds} = c \tag{3.5}$$

subject to the initial conditions (3.4), one can then obtain a unique local solution for the original system (3.3)-(3.4), provided that the Jacobian

$$J = \det \begin{bmatrix} dx/ds & dy/ds \\ dx/dx_0 & dy/dx_0 \end{bmatrix}$$

is nonzero along the initial curve (3.4). In fact, the solvability condition $J \neq 0$ is equivalent to the condition that the initial curve (3.4) is not itself a characteristic curve.

Example 3. We use the method of characteristics to solve the problem

$$xu_x + yu_y = 2u, \qquad u(x, 1) = f(x).$$

In this case, the characteristic equations are

$$\frac{dx}{ds} = x, \qquad \frac{dy}{ds} = y, \qquad \frac{du}{ds} = 2u$$

so we can easily solve them to get

$$x = x_0 e^s, \qquad y = y_0 e^s, \qquad u = u_0 e^{2s}.$$

In view of our initial condition $u(x_0, 1) = f(x_0)$, this gives

$$y = e^s \quad \Longrightarrow \quad u = u_0 e^{2s} = f(x_0)y^2 = f(x/y)y^2.$$

Example 4. We use the method of characteristics to show that the problem

$$u_x + u_y = 1, \qquad u(x, x) = 1$$

has no solutions. In this case, the characteristic equations are $x' = y' = u' = 1$ and so

$$x = s + x_0, \qquad y = s + y_0, \qquad u = s + u_0.$$

Since the line $y = x$ is one of the characteristic curves, it is better to avoid it and impose some other initial condition. Using a generic one such as $u(x_0, 0) = f(x_0)$, we find

$$y = s = x - x_0 \quad \Longrightarrow \quad u = s + f(x_0) = y + f(x - y),$$

so all solutions of the PDE must have this form. In particular, $u(x, x) = x + f(0)$ cannot be equal to a constant and the given initial condition is never satisfied.