Useful calculus facts

Definition 1. The gradient of a scalar function $u\colon \mathbb{R}^n \to \mathbb{R}$ is defined as the vector

$$\nabla u = \langle u_{x_1}, u_{x_2}, \ldots, u_{x_n} \rangle.$$

The divergence of a vector-valued function $\boldsymbol{F}\colon \mathbb{R}^n \to \mathbb{R}^n$ is defined as the scalar

$$\nabla \cdot \boldsymbol{F} = \sum_{i=1}^{n} \frac{\partial \boldsymbol{F}_i}{\partial x_i}\,.$$

Finally, the Laplacian of a scalar function $u\colon \mathbb{R}^n \to \mathbb{R}$ is defined as the scalar

$$\Delta u = \nabla \cdot \nabla u = \sum_{i=1}^{n} u_{x_i x_i}.$$

Theorem 2 (Divergence theorem). If the vector-valued function $\boldsymbol{F}$ is $\mathcal{C}^1$, then

$$\int_A \nabla \cdot \boldsymbol{F}(x)\,dx = \int_{\partial A} \boldsymbol{F}(x) \cdot \boldsymbol{n}\,dS,$$

where ∂A denotes the boundary of the set A and $\boldsymbol{n}$ is the outward unit normal vector.

Theorem 3 (Green's identities). If $u\colon \mathbb{R}^n \to \mathbb{R}$ is $\mathcal{C}^1$ and $v\colon \mathbb{R}^n \to \mathbb{R}$ is $\mathcal{C}^2$, then

$$\int_A u \Delta v\,dx = -\int_A \nabla u \cdot \nabla v\,dx + \int_{\partial A} u \nabla v \cdot \boldsymbol{n}\,dS.$$

If u, v are both $\mathcal{C}^2$, then

$$\int_A (u\Delta v - v\Delta u)\,dx = \int_{\partial A} (u\nabla v - v\nabla u) \cdot \boldsymbol{n}\,dS.$$

Theorem 4 (Polar coordinates). An integral over a ball can be expressed in terms of a surface integral over a sphere. Given any $x \in \mathbb{R}^n$ and any $r > 0$, that is, one has

$$\int_{|y-x|\le r} u(y)\,dy = \int_0^r \int_{|y-x|=t} u(y)\,dS_y\,dt.$$