Laplace equation: Poisson formulas

Theorem 1 (Poisson formula for the upper half space). Suppose g is continuous and bounded in the upper half space $A \subset \mathbb{R}^n$. Then the Dirichlet problem

$$\Delta u(x) = 0 \quad \text{in } A, \qquad u(x) = g(x) \quad \text{on } \partial A$$

has a unique bounded solution which is given by the formula

$$u(x) = \frac{2x_n}{n\alpha_n} \int_{\partial A} \frac{g(y)}{|x-y|^n} \, dS_y.$$

And if $A \subset \mathbb{R}^2$ is the upper half plane, then this formula gives

$$u(x,y) = \frac{y}{\pi} \int_{-\infty}^{\infty} \frac{g(z,0)}{(x-z)^2 + y^2} \, dz.$$

Theorem 2 (Poisson formula for the unit ball). Suppose g is continuous and bounded in the unit ball $B \subset \mathbb{R}^n$. Then the Dirichlet problem

$$\Delta u(x) = 0 \quad \text{in } B, \qquad u(x) = g(x) \quad \text{on } \partial B$$

has a unique solution which is given by the formula

$$u(x) = \frac{1 - |x|^2}{n\alpha_n} \int_{\partial B} \frac{g(y)}{|x-y|^n} \, dS_y.$$

Remark. In the theorems above, the boundary condition is meant to be interpreted as a limit. In the first case, for instance, one has $u(x) \to g(x)$ as $x_n \to 0$.

Theorem 3 (Dirichlet principle). Suppose $A \subset \mathbb{R}^n$ is bounded and $g\colon \mathbb{R}^n \to \mathbb{R}$ a given function. Out of all functions that satisfy $u(x) = g(x)$ on ∂A, the one that minimizes

$$I(u) = \int_A |\nabla u|^2 \, dx$$

is the one which is harmonic within A.