Laplace equation: general facts

Definition 1 (Harmonic). A function $u \colon \mathbb{R}^n \to \mathbb{R}$ is called harmonic, if $\Delta u = 0$.

Theorem 2 (Mean value property over spheres). The average value of a harmonic function u over a sphere is equal to its value at the centre. In other words, one has

$$u(x) = \frac{1}{n\alpha_n r^{n-1}} \int_{|y-x|=r} u(y)\, dS_y$$

for all $x \in \mathbb{R}^n$ and all $r > 0$, where α_n denotes the volume of the unit ball in $\mathbb{R}^n$.

Remark. If α_n denotes the volume of the unit ball in $\mathbb{R}^n$, then $\alpha_n r^n$ gives the volume of a ball of radius r, while $n\alpha_n r^{n-1}$ gives the surface of a sphere of radius r.

Theorem 3 (Maximum principle). If u is harmonic in a closed, bounded region, then the min/max values of u are attained on the boundary.

Theorem 4 (Uniqueness of solutions). Consider the most general Dirichlet problem

$$\Delta u(x) = f(x) \quad \text{in } A, \qquad u(x) = g(x) \quad \text{on } \partial A$$

over a bounded region $A \subset \mathbb{R}^n$. Then this problem admits at most one solution.

Theorem 5 (Smoothness). Every harmonic function is smooth.

Theorem 6 (Liouville's theorem). All bounded harmonic functions are constant.

Theorem 7 (Fundamental solution). Consider the function $F \colon \mathbb{R}^n \to \mathbb{R}$ defined by

$$F(x) = \left\{ \begin{array}{cc} -\dfrac{\log |x|}{2\pi} & \text{if } n = 2 \\[2ex] \dfrac{|x|^{2-n}}{n(n-2)\alpha_n} & \text{if } n \geq 3 \end{array} \right\}.$$

Then $\Delta F(x) = 0$ at all points $x \neq 0$ and also $-\Delta F = \delta$ in the sense of distributions.