Burgers' equation

Theorem 1 (Rankine-Hugoniot). If u is a weak solution of $u_t + F(u)_x = 0$ which is discontinuous across the curve $x = h(t)$ but smooth on either side of the curve, then

$$h'(t) = \frac{F(u^+) - F(u^-)}{u^+ - u^-},$$

where u^- and u^+ are the limits of $u(x,t)$ as x approaches $h(t)$ from the left and the right, respectively. In the case of Burgers' equation $u_t + uu_x = 0$, this condition becomes

$$h'(t) = \frac{u^+ + u^-}{2}.$$

Definition 2 (Shock). A shock solution of the equation $u_t + F(u)_x = 0$ is a weak solution which satisfies the entropy condition

$$F'(u^-) \geq h'(t) \geq F'(u^+)$$

across any curve $x = h(t)$ of discontinuity. In the case of Burgers' equation $u_t + uu_x = 0$, this condition is equivalent to the condition $u^- \geq u^+$.

Theorem 3 (Burgers' equation). The solution of the Burgers' equation

$$u_t + uu_x = 0, \qquad u(x,0) = f(x)$$

is given implicitly by the formula $u = f(x - ut)$, where $x - ut = x_0$ along characteristics.