# Second-order linear equations

**Theorem 1** (**Canonical forms**). Consider a second-order linear PDE, say

$$Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = G.$$

If the discriminant $\Delta = B^2 - 4AC$ is positive, then the PDE is called hyperbolic and there is a linear change of variables $(x, y) \to (v, w)$ that transforms it into a PDE of the form

$$u_{vv} - u_{ww} = f(u_v, u_w, u, v, w).$$

Here, the left hand side is very simple, while the right hand side contains only lower-order terms. If $\Delta < 0$, then the PDE is called elliptic and can be similarly transformed into

$$u_{vv} + u_{ww} = f(u_v, u_w, u, v, w).$$

If $\Delta = 0$, finally, then the PDE is called parabolic and it can be transformed into

$$u_{vv} = f(u_v, u_w, u, v, w).$$

**Example 2.** To find all solutions of the equation $u_{tt} - u_{xt} - 2u_{xx} = 0$, we write

$$0 = (\partial_t^2 - \partial_x\partial_t - 2\partial_x^2)u = (\partial_t + \partial_x)(\partial_t - 2\partial_x)u.$$

Suppose we can find variables $v, w$ such that $\partial_v = \partial_t + \partial_x$ and $\partial_w = \partial_t - 2\partial_x$. Then

$$0 = \partial_v\partial_w u \quad \Longrightarrow \quad \partial_w u = F_1(w) \quad \Longrightarrow \quad u = F_2(w) + F_3(v)$$

so all solutions must have this form. To find the variables $v$ and $w$, we now note that

$$\partial_v = t_v\partial_t + x_v\partial_x, \qquad \partial_w = t_w\partial_t + x_w\partial_x$$

by the chain rule, while $\partial_v = \partial_t + \partial_x$ and $\partial_w = \partial_t - 2\partial_x$ by above. Thus, we need to have

$$t_v = x_v = t_w = 1, \qquad x_w = -2$$

so we can simply let $t = v + w$ and $x = v - 2w$. Solving for $v$ and $w$, we conclude that

$$t - x = 3w, \qquad 2t + x = 3v \quad \Longrightarrow \quad u = F(t - x) + G(2t + x).$$