PDEs, 2009 exam
Solutions

1a. It is first-order, linear and homogeneous.

1b. Using the method of characteristics, we get the system of ODEs

$$\frac{dt}{ds} = t - 1, \qquad \frac{dx}{ds} = -x, \qquad \frac{du}{ds} = 0$$

subject to the initial condition $u(0, x_0) = f(x_0)$. This gives

$$u = u_0 = f(x_0), \qquad x = x_0 e^{-s}, \qquad \log|t - 1| = s$$

and we can eliminate s, x_0 to conclude that

$$e^s = |t - 1| \quad \Longrightarrow \quad u = f(xe^s) = f(x|t - 1|).$$

1c. The solution found in part (b) is certainly global. Now, the characteristic curves

$$x|t - 1| = xe^s = x_0$$

are hyperbolas which lie either above the line $t = 1$ or below it. Since the initial data are given at time $t = 0$, a unique solution exists only for times $t < 1$.

2a. The unique solution is given by the formula

$$u(t, x) = \frac{1}{\sqrt{4k\pi t}} \int_{-\infty}^{\infty} \exp\left(-\frac{(x - y)^2}{4kt}\right) \cos(\omega y)\, dy$$

and is thus the real part of

$$w(t, x) = \frac{1}{\sqrt{4k\pi t}} \int_{-\infty}^{\infty} \exp\left(-\frac{(x - y)^2}{4kt}\right) e^{i\omega y}\, dy.$$

Integrals of this form were computed in Homework 3, Problem 5 and one has

$$\frac{1}{\sqrt{4k\pi t}} \int_{-\infty}^{\infty} \exp\left(-\frac{(x - y)^2}{4kt}\right) e^{ay}\, dy = e^{a^2kt + ax}.$$

Once we now combine all these facts, we find that

$$u(t, x) = \operatorname{Re} w(t, x) = \operatorname{Re} e^{-\omega^2 kt + i\omega x} = e^{-\omega^2 kt} \cos(\omega x).$$

2b. The unique solution is given by the formula

$$u(t,x) = \sum_{n=-\infty}^{\infty} \int_{nL}^{(n+1)L} S(x-y,t) \cdot f(y)\,dy.$$

Using the substitution $z = y - nL$, one then easily finds that

$$\begin{aligned} u(t,x) &= \sum_{n=-\infty}^{\infty} \int_0^L S(x-z-nL,t) \cdot f(z+nL)\,dz \\ &= \sum_{n=-\infty}^{\infty} \int_0^L S(x-z-nL,t) \cdot f(z)\,dz. \end{aligned}$$

This already gives the desired integral representation with

$$k(t,x,y) = \sum_{n=-\infty}^{\infty} S(x-y-nL,t).$$

4a. The even extensions of the given initial data to the whole real line are

$$\varphi(x) = 0, \qquad \psi(x) = |x|.$$

Thus, the corresponding solution of the problem on the whole real line is

$$u(t,x) = \frac{1}{2c} \int_{x-ct}^{x+ct} |s|\,ds.$$

When $x > ct$, this formula gives

$$u(t,x) = \frac{1}{2c} \int_{x-ct}^{x+ct} s\,ds = \frac{(x+ct)^2 - (x-ct)^2}{4c} = xt.$$

When $x < ct$, on the other hand, it gives

$$u(t,x) = \frac{1}{2c} \int_0^{x+ct} s\,ds - \frac{1}{2c} \int_{x-ct}^0 s\,ds = \frac{x^2 + (ct)^2}{2c}.$$

4b. It is a weak solution and a distributional solution but not a classical solution. For instance, $u_{tt} = 0$ whenever $x > ct$, whereas $u_{tt} = c$ whenever $x < ct$.

4c. Homework 2, Problem 1 is very similar.

5a. See Homework 4, Problem 6. I will not ask for any formulas.

5b. The boundary condition should be interpreted as the limit when $y \to 0$.

5c. This is very similar to result 16 from the list of our main results.

7a. See the solutions to the 2007 exam, Problem 8a.

7b. It can only be a classical solution, if it is continuous, namely if $a = b$. To be a weak solution, it must satisfy the Rankine-Hugoniot condition $c = \frac{a+b}{2}$. As for the entropy condition, this requires that $a \geq b$.