UNIVERSITY OF DUBLIN

XMA4191

TRINITY COLLEGE

Faculty of Engineering, Mathematics
and Science

school of mathematics

JS Maths
SS Maths

Trinity Term 2009

Course 419

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ATTEMPT FIVE QUESTIONS
All questions have equal weight (20 points)

1. (a) *(4 points)* For the differential equation

$$(t-1)\frac{\partial u}{\partial t} - x\frac{\partial u}{\partial x} = 0$$

state its order, whether it is linear or nonlinear, and if linear whether it is homogeneous or inhomogeneous.

(b) *(10 points)* Solve the initial value problem

$$(t-1)\frac{\partial u}{\partial t} - x\frac{\partial u}{\partial x} = 0 \qquad u(0,x) = f(x)$$

(c) *(6 points)* What can you say about existence and uniqueness of classical solutions of this differential equation?

2. (a) *(10 points)* Solve the initial value problem for the Diffusion Equation

$$\frac{\partial u}{\partial t} - k\frac{\partial^2 u}{\partial x^2} = 0 \qquad u(0,x) = \cos(\omega x)$$

(b) *(10 points)* More generally, the Diffusion Equation with initial data f,

$$\frac{\partial u}{\partial t} - k\frac{\partial^2 u}{\partial x^2} = 0 \qquad u(0,x) = f(x)$$

with f is periodic with period L the solution is of the form

$$u(t,x) = \int_0^L k(t,x,y) f(y)\, dy.$$

Find k.

3. (a) *(10 points)* Suppose that u satisfies the initial value problem for the Diffusion Equation with initial data f,

$$\frac{\partial u}{\partial t} - k\frac{\partial^2 u}{\partial x^2} = 0 \qquad u(0,x) = f(x).$$

For which p is it true that if $f \in L^p(\mathbf{R})$ then

$$\lim_{t\to 0^+} u(t,\cdot) = f$$

with convergence in $L^p(\mathbf{R})$?

(b) *(10 points)* Justify your answer from the previous part. In other words, if you have said that the statement is false for some p then give an f for which it fails. If you have said that it holds for some p then say from what theorem this follows.

4. (a) *(8 points)* Solve the initial value problem for the Wave Equation on the half-line $x \geq 0$ with Neumann boundary conditions and the given initial data,

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0 \qquad u(0,x) = 0 \qquad \frac{\partial u}{\partial t}(0,x) = x \qquad \frac{\partial u}{\partial x}(t,0) = 0.$$

(b) *(6 points)* What kind of solution have you found: classical? weak? distribution?

(c) *(6 points)* Prove that if u solves the initial value problem

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = h(t,x) \quad u(0,x) = f(x) \quad \frac{\partial u}{\partial t}(0,x) = g(x)$$

for the inhomogeneous Wave Equation in $\mathbf{R}$ with f, g and h all odd functions of x then u is also an odd function of x.

5. (a) *(5 points)* State the Poisson formula for the disc of radius a centred at 0. You may give any of the various equivalent forms.

(b) *(5 points)* The Poisson formula for the upper half plane takes the form

$$u(x,y) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{y f(z)}{(x-z)^2 + y^2} dz.$$

This is supposed to solve the Dirichlet problem for the Laplace equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad u(x,0) = f(x),$$

and yet, when we substitute 0 for y in the Poisson formula we get

$$u(x,0) = 0.$$

Explain.

(c) *(10 points)* Prove that

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \delta$$

as distributions on $\mathbf{R}^2$, where

$$u(x,y) = -\frac{1}{4\pi} \log(x^2 + y^2)$$

6. (a) *(6 points)* State Green's first and second identities.

   (b) *(8 points)* Prove that any two solutions of the Neumann problem with the same boundary data on a bounded connected domain with smooth boundary differ by a constant.

   (c) *(6 points)* Show, by means of an example, that this statement may fail for unbounded domains.

7. (a) *(10 points)* Solve the initial value problem for Burgers' Equation with the given initial data,

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0 \qquad u(0,x) = x^2.$$

   Be sure to state where your solution is defined.

   (b) *(10 points)* For which a, b, c is

$$u(t,x) = \begin{cases} a & \text{if } x < ct \\ b & \text{if } x \geq ct \end{cases}$$

   a classical solution of Burgers' Equation? For which is it a weak solution? For which does it satisfy the entropy condition?