PDEs, 2007 exam
Solutions

1a. Using the method of characteristics, we get the system of ODEs

$$\frac{dt}{ds} = t - a, \qquad \frac{dx}{ds} = 1, \qquad \frac{du}{ds} = 0$$

subject to the initial condition $u(0, x_0) = f(x_0)$. This gives

$$u = u_0 = f(x_0), \qquad x = s + x_0, \qquad \log|t - a| = s + \log|a|$$

and we can eliminate s, x_0 to conclude that

$$s = \log|t/a - 1| \quad \Longrightarrow \quad u = f(x - s) = f(x - \log|t/a - 1|).$$

1b. The solution exists for all times for which $t/a - 1$ remains nonzero. If $a < 0$, it exists for all times $t \geq 0$, so the solution is global. If $a > 0$, it exists for all times $0 \leq t < a$, so the solution is only local. In either case, a unique solution exists by part (a).

2a. When $\varphi = 0$, the solution is given by d'Alembert's formula

$$u(t, x) = \frac{1}{2c} \int_{x-ct}^{x+ct} \psi(s)\, ds$$

and we now need to express this formula as a convolution

$$u(t, x) = K(t, x) * \psi(x) = \int_{-\infty}^{\infty} K(t, x - s) \cdot \psi(s)\, ds.$$

Comparing these two equations, we see that K must be given by

$$K(t, x - s) = \left\{ \begin{array}{cc} \frac{1}{2c} & \text{if } x - ct \leq s \leq x + ct \\ 0 & \text{otherwise} \end{array} \right\}.$$

In other words, K must be given by

$$K(t, y) = \left\{ \begin{array}{cc} \frac{1}{2c} & \text{if } |y| \leq ct \\ 0 & \text{otherwise} \end{array} \right\}.$$

2b. Using d'Alembert's formula with $c = 1$, we get

$$u(t, x) = \frac{1}{2} \int_{x-t}^{x+t} \frac{ds}{1 + s^2} = \frac{\arctan(x + t) - \arctan(x - t)}{2}.$$

3a. The unique solution is given by Duhamel's formula

$$u(t,x) = \frac{1}{2c}\int_{x-ct}^{x+ct} |s|\, ds + \frac{1}{2c}\int_0^t \int_{x-c(t-\tau)}^{x+c(t-\tau)} dy\, d\tau.$$

When it comes to the rightmost integral, one easily finds that

$$\frac{1}{2c}\int_0^t \int_{x-c(t-\tau)}^{x+c(t-\tau)} dy\, d\tau = \int_0^t (t-\tau)\, d\tau = \frac{t^2}{2}.$$

To compute the other integral, we have to distinguish cases. If $x - ct > 0$, then

$$\frac{1}{2c}\int_{x-ct}^{x+ct} |s|\, ds = \frac{1}{2c}\int_{x-ct}^{x+ct} s\, ds = \frac{(x+ct)^2 - (x-ct)^2}{4c} = xt.$$

If $x + ct < 0$, then

$$\frac{1}{2c}\int_{x-ct}^{x+ct} |s|\, ds = -\frac{1}{2c}\int_{x-ct}^{x+ct} s\, ds = -xt.$$

And if $x - ct < 0 < x + ct$, then

$$\frac{1}{2c}\int_{x-ct}^{x+ct} |s|\, ds = \frac{1}{2c}\int_0^{x+ct} s\, ds - \frac{1}{2c}\int_{x-ct}^{0} s\, ds = \frac{x^2 + (ct)^2}{2c}.$$

3b. It is a weak solution and a distributional solution but not a strong (classical) solution. For instance, $u_t(0,x) = |x|$ is continuous but not differentiable.

3c. Homework 2, Problem 1 is almost identical.

4a. The unique solution is given by the formula

$$\begin{aligned} u(t,x) &= \frac{1}{\sqrt{4k\pi t}}\int_{-\infty}^{\infty} \exp\left(-\frac{(x-y)^2}{4kt}\right) \sinh y\, dy \\ &= \frac{1}{2\sqrt{4k\pi t}}\int_{-\infty}^{\infty} \exp\left(-\frac{(x-y)^2}{4kt}\right) e^y\, dy \\ &\quad - \frac{1}{2\sqrt{4k\pi t}}\int_{-\infty}^{\infty} \exp\left(-\frac{(x-y)^2}{4kt}\right) e^{-y}\, dy. \end{aligned}$$

Integrals of this form were computed in Homework 3, Problem 5 and one has

$$\frac{1}{\sqrt{4k\pi t}}\int_{-\infty}^{\infty} \exp\left(-\frac{(x-y)^2}{4kt}\right) e^{ay}\, dy = e^{a^2kt + ax}$$

for any $a \in \mathbb{R}$ whatsoever. Once we now combine the last two equations, we get

$$u(t,x) = \frac{e^{kt+x}}{2} - \frac{e^{kt-x}}{2} = e^{kt}\sinh x.$$

4b. Extend φ to the whole real line in such a way that the extension is odd and periodic of period $2L$. This can be done by setting

$$\varphi_{\text{ext}}(x) = \left\{ \begin{array}{cc} \varphi(x-2nL) & \text{if } 2nL \leq x \leq (2n+1)L \\ -\varphi(2nL+2L-x) & \text{if } (2n+1)L < x < 2(n+1)L \end{array} \right\},$$

where $n \in \mathbb{Z}$ runs through all integers. The solution on the real line is then

$$u(t,x) = \sum_{n=-\infty}^{\infty} \int_{2nL}^{2(n+1)L} S(x-y,t) \cdot \varphi_{\text{ext}}(y)\, dy$$

and we can use the definition of φ_{ext} to get

$$\begin{aligned} u(t,x) = & \sum_{n=-\infty}^{\infty} \int_{2nL}^{(2n+1)L} S(x-y,t) \cdot \varphi(y-2nL)\, dy \\ & - \sum_{n=-\infty}^{\infty} \int_{(2n+1)L}^{2(n+1)L} S(x-y,t) \cdot \varphi(2nL+2L-y)\, dy. \end{aligned}$$

Changing variables to simplify the two integrals, we conclude that

$$\begin{aligned} u(t,x) = & \sum_{n=-\infty}^{\infty} \int_{0}^{L} S(x-z-2nL,t) \cdot \varphi(z)\, dz \\ & - \sum_{n=-\infty}^{\infty} \int_{0}^{L} S(x+z-2nL-2L,t) \cdot \varphi(z)\, dz. \end{aligned}$$

This is one way to obtain an integral representation for the solution. A second and slightly harder way would be to use separation of variables and Fourier series.

5a. See result 13 from the list of our main results.

5b. See Homework 4, Problem 8.

5c. See result 16 from the list of our main results.

6a. See Homework 4, Problem 10.

6b. No, it is not unique anymore. If u is a solution, then so is $u + ay$ for any $a \in \mathbb{R}$.

6c. The boundary condition should be interpreted as the limit when $r \to a$.

8a. First of all, we use the standard formula $u = f(x-ut)$ to get

$$u = f(x-ut) = (x-ut)^2 \quad \Longrightarrow \quad t^2u^2 - (2tx+1)u + x^2 = 0.$$

This quadratic equation has two solutions which are given by

$$u = \frac{2tx + 1 \pm \sqrt{4tx+1}}{2t^2}.$$

Since the denominator vanishes when $t = 0$, the numerator must also do, hence

$$u = \frac{2tx + 1 - \sqrt{4tx+1}}{2t^2}.$$

8b. A shock is a weak solution which satisfies both the Rankine-Hugoniot condition and the entropy condition along any curve of discontinuity.

8c. Fix real numbers a, b, c and consider the function

$$u(x,t) = \left\{ \begin{array}{ll} a & \text{if } x < ct \\ b & \text{if } x \geq ct \end{array} \right\}.$$

This is a shock solution whenever $c = \frac{a+b}{2}$ and $a \geq b$. If $c = \frac{a+b}{2}$ and $a < b$, then it is a weak solution which fails to satisfy the entropy condition.