UNIVERSITY OF DUBLIN

XMA4191

TRINITY COLLEGE

Faculty of Engineering, Mathematics
and Science

school of mathematics

JS Maths
SS Maths

Trinity Term 2007

Course 419

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ATTEMPT SIX QUESTIONS
All questions have equal weight (20 points)

1. (a) *(14 points)* For $a \neq 0$, solve the initial value problem

$$(t-a)u_t + u_x = 0 \qquad u(0,x) = f(x).$$

   (b) *(6 points)* What can you say about the existence (local and global) and uniqueness of solutions to this problem? Your answer may depend on a.

2. (a) *(8 points)* The solution of the initial value problem for the homogeneous Wave Equation

$$u_{tt} - c^2 u_{xx} = 0 \qquad u|_{t=0} = \varphi \qquad u_t|_{t=0} = \psi$$

   in the special case $\varphi = 0$ can be written in the form

$$u(t,\cdot) = K(t,\cdot) \star \psi$$

   for an appropriate function K. What is this K?

   (b) *(12 points)* Solve the initial value problem for the Wave Equation

$$u_{tt} - u_{xx} = 0$$

   with initial data

$$u(0,x) = 0 \qquad u_t(0,x) = \frac{1}{1+x^2}.$$

3. For the initial value problem for the *inhomogeneous* Wave Equation

$$u_{tt} - c^2 u_{xx} = 1 \qquad u(0,x) = 0 \qquad u_t(0,x) = |x|$$

   (a) *(12 points)* Find the solution.

   (b) *(4 points)* Is the solution you found a strong solution, weak solution, distribution solution?

   (c) *(4 points)* Prove that if

$$u_{tt} - c^2 u_{xx} = f \qquad u(0,x) = \varphi(x) \qquad u_t(0,x) = \psi(x)$$

   where f, φ and ψ are even functions of x, then u is also an even function of x.

4. (a) *(10 points)* Solve the initial value problem for the Diffusion Equation on the real line

$$u_t - ku_{xx} = 0 \quad u(0,x) = \sinh(x)$$

(b) *(10 points)* Derive an integral representation for solutions of the Dirichlet problem for the Diffusion Equation on the interval $[0, L]$,

$$u_t - ku_{xx} = 0 \quad u(0,x) = \varphi(x) \qquad u(t,0) = u(t,L) = 0$$

You may assume the corresponding formula for the Dirichlet problem on the real line.

5. (a) *(5 points)* State Green's First and Second Identities.

(b) *(5 points)* Prove that the Dirichlet problem for the homogeneous Laplace Equation in a bounded domain has at most one solution. You may take as given Green's Identities or the Maximum Principle.

(c) *(10 points)* Consider the function

$$u = \frac{1}{4\pi} \frac{1}{\sqrt{x^2 + y^2 + z^2}}$$

as a distribution in the usual way, *i.e.*

$$\langle u, \varphi \rangle = \int_{\mathbf{R}^3} u\varphi.$$

Prove that

$$u_{xx} + u_{yy} + u_{zz} = \delta$$

in the sense of distributions.

6. (a) *(10 points)* Solve the Dirichlet problem for the homogeneous Laplace Equation in the upper half plane $y \geq 0$ for a *bounded* u with boundary values

$$u(x,0) = \begin{cases} -1 & \text{if } x < 0 \\ +1 & \text{if } x > 0 \end{cases}$$

(b) *(5 points)* What if the restriction to bounded u is removed? Is the solution from the previous part still unique? Justify your answer.

(c) *(5 points)* Many books on Partial Differential Equations claim that the solution of the Dirichlet problem

$$u(a\cos\theta, a\sin\theta) = f(\theta)$$

for the Laplace Equation

$$u_{xx} + u_{yy} = 0$$

in the disc

$$x^2 + y^2 \leq a^2$$

is given by Poisson's Formula

$$u(r\cos\theta, r\sin\theta) = \frac{1}{\pi}\int_{-\pi}^{\pi} \frac{a^2 - r^2}{a^2 - 2ar\cos(\theta - \varphi) + r^2} f(\varphi)\, d\varphi.$$

Substituting $r = a$ into Poisson's Formula, however, appears to give

$$u(a\cos\theta, a\sin\theta) = 0$$

rather than

$$u(a\cos\theta, a\sin\theta) = f(\theta)$$

since the $r^2 - a^2$ factor makes the integrand zero. Explain in what sense, if any, the Poisson Formula solves the Dirichlet Problem.

7. (a) *(4 points)* Prove that there is a symmetry of the two dimensional Laplace Equation which cyclically permutes the x-axis, the y-axis and the unit circle.

   (b) *(10 points)* Find the symmetry described in the previous part explicitly if you haven't already done so. There are, in fact, several possible choices. Any of them will do.

   (c) *(6 points)* The symmetry has two fixed points. Find them.

8. (a) *(12 points)* Solve Burgers' Equation

$$u_t + uu_x = 0$$

   in the region

$$4tx + 1 \geq 0$$

   with initial data

$$u(0, x) = x^2.$$

   (b) *(4 points)* What is meant by a shock, in the context of Burgers' Equation?

   (c) *(4 points)* Give an example of a shock solution and an example of a weak solution which fails to satisfy the entropy condition.

9. (a) *(6 points)* Prove that all solutions of the Klein-Gordon Equation

$$u_{tt} - u_{xx} + u = 0$$

satisfy the Energy Conservation Law

$$\int_{-\infty}^{+\infty} (u_t^2 + u_x^2 + u^2)\, dx = \int_{-\infty}^{+\infty} (\psi^2 + \varphi_x^2 + \varphi^2)\, dx$$

where

$$u(0, x) = \varphi(x) \qquad u_t(0, x) = \psi(x).$$

Note: Beyond the fact that all the derivatives which appear exist and are continuous and that the integral $\int_{-\infty}^{+\infty} (\psi^2 + \varphi_x^2 + \varphi^2)\, dx$ exists, no further hypotheses on u, φ and ψ are needed. If your proof requires additional hypotheses, be sure to state them clearly.

(b) *(6 points)* Assuming the result of the previous part, even if you didn't succeed in proving it, show that the initial value problem

$$u_{tt} - u_{xx} + u = 0 \qquad u(0, x) = \varphi(x) \qquad u_t(0, x) = \psi(x)$$

has a unique solution.

(c) *(2 points)* State Young's Inequality.

(d) *(6 points)* The unique solution referred to previously is, in fact,

$$u(t, x) = \int_{x-t}^{x+t} K_t(t, x - y)\varphi(y)\, dy + \int_{x-t}^{x+t} K(t, x - y)\psi(y)\, dy$$

where

$$K(t, x) = \frac{1}{2} J_0(\sqrt{t^2 - x^2})$$

and J_0 is the Bessel function of order 0. All you need to know about J_0 for this part is that J_0 is bounded. Given this, prove that for any $1 \le r \le p \le \infty$ there is a constant $C_{p,r}$ such that if $\varphi = 0$ and $\psi \in L^p(\mathbf{R})$ then

$$\|u(t, \cdot)\|_{L^r(\mathbf{R})} \le C_{p,r} t^{1+\frac{1}{r}-\frac{1}{p}} \|\psi\|_{L^p(\mathbf{R})}.$$