Sample final exam

1a. Indeed, every solution has the form $x(t) = c_1 \sin t + c_2 \cos t$.

1b. By the method of undetermined coefficients, every solution has the form

$$x(t) = c_1 \sin t + c_2 \cos t + At \sin t + Bt \cos t$$

for some arbitrary constants c_1, c_2. The constants A, B can be determined, if needed, but that is not necessary here. Since $x(t)$ does not satisfy the homogeneous equation, either A or B is nonzero, and this already implies that $x(t)$ is unbounded.

1c. Indeed, every solution has the form $x(t) = c_1 \sin t + c_2 \cos t + A \sin(2t) + B \cos(2t)$.

1d. Setting $t = 0$ in the ODE, we get $x(0) = 0$, and this violates the initial condition.

1e. Using separation of variables, one easily finds that

$$\frac{dx}{dt} = \frac{x}{t} \quad \Longrightarrow \quad \int \frac{dx}{x} = \int \frac{dt}{t} \quad \Longrightarrow \quad \log|x| = \log|t| + C \quad \Longrightarrow \quad x = Ct.$$

Strictly speaking, this computation can only be justified at points where x, t are both nonzero. The easiest way to get around this difficulty is to simply check that $x = Ct$ is, in fact, a solution to the initial value problem.

2a. Since $y \neq 0$ by uniqueness, we may separate variables to get

$$\begin{aligned} \frac{dy}{dt} = -2ty \quad &\Longrightarrow \quad \int \frac{dy}{y} = -\int 2t\, dt \\ &\Longrightarrow \quad \log|y| = -t^2 + C \quad \Longrightarrow \quad y = Ce^{-t^2}. \end{aligned}$$

To ensure that $y(0) = e$, we need to have $C = e$ and this implies $y = e^{1-t^2}$.

2b. Noting that the left hand side of the ODE is a perfect derivative, we get

$$\left(y' - \frac{1}{t} \cdot y\right)' = t \log t \quad \Longrightarrow \quad y' - \frac{1}{t} \cdot y = \int t \log t\, dt.$$

In order to compute the integral, we now integrate by parts to find that

$$\int t \log t\, dt = \frac{t^2}{2} \cdot \log t - \int \frac{t^2}{2} \cdot \frac{1}{t}\, dt = \frac{t^2 \log t}{2} - \frac{t^2}{4} + C_1. \tag{1}$$

In particular, it remains to solve the first-order linear ODE

$$y' - \frac{1}{t} \cdot y = \frac{t^2 \log t}{2} - \frac{t^2}{4} + C_1.$$

Note that an integrating factor for this ODE is given by

$$\mu = \exp\left(-\int \frac{1}{t}\,dt\right) = \exp(-\log t) = t^{-1}.$$

Multiplying by this factor and using equation (1), we conclude that

$$\begin{aligned}
\left(\frac{y}{t}\right)' = \frac{t\log t}{2} - \frac{t}{4} + \frac{C_1}{t} \quad &\Longrightarrow \quad \frac{y}{t} = \int \frac{t\log t}{2} - \frac{t}{4} + \frac{C_1}{t}\,dt \\
&\Longrightarrow \quad \frac{y}{t} = \frac{t^2\log t}{4} - \frac{t^2}{8} - \frac{t^2}{8} + C_1\log t + C_2 \\
&\Longrightarrow \quad y = \frac{t^3\log t}{4} - \frac{t^3}{4} + C_1 t\log t + C_2 t.
\end{aligned}$$

To ensure that $y(1) = 0$, we need to have $C_2 = 1/4$. Once we now note that

$$y' = \frac{3t^2\log t}{4} + \frac{t^2}{4} - \frac{3t^2}{4} + C_1\log t + C_1 + C_2,$$

the initial condition $y'(1) = 0$ imposes the additional restriction

$$0 = \frac{1}{4} - \frac{3}{4} + C_1 + \frac{1}{4} \quad \Longrightarrow \quad C_1 = \frac{1}{4}.$$

In other words, we need to have $C_1 = C_2 = 1/4$, and this finally gives

$$y = \frac{t^3\log t - t^3 + t\log t + t}{4}.$$

3a. When $y_1 = t^{-1}$, we have $y_1' = -t^{-2}$ and also $y_1'' = 2t^{-3}$, so

$$t^2 y_1'' + 3t y_1' + y_1 = 2t^{-1} - 3t^{-1} + t^{-1} = 0$$

and $y_1 = t^{-1}$ is a solution, indeed. We now use reduction of order to find a second solution of the form $y_2 = y_1 v = t^{-1}v$. Differentiating, we get

$$y_2 = t^{-1}v, \qquad y_2' = -t^{-2}v + t^{-1}v', \qquad y_2'' = 2t^{-3}v - 2t^{-2}v' + t^{-1}v''$$

and thus $y_2 = t^{-1}v$ is also a solution, provided that

$$0 = t^2 y_2'' + 3t y_2' + y_2 = tv'' + v' \quad \Longrightarrow \quad (tv')' = 0.$$

Integrating the last equation and then integrating again, we conclude that

$$v' = C_1/t \quad \Longrightarrow \quad v = C_1\log t + C_2 \quad \Longrightarrow \quad y_2 = C_1 t^{-1}\log t + C_2 t^{-1}.$$

3b. To find the homogeneous solution y_h, we note that

$$\begin{aligned} \lambda^3 + \lambda^2 - 4\lambda - 4 = 0 \quad &\Longrightarrow \quad \lambda^2(\lambda+1) - 4(\lambda+1) = 0 \\ &\Longrightarrow \quad (\lambda+1)(\lambda-2)(\lambda+2) = 0 \\ &\Longrightarrow \quad y_h = c_1 e^{-t} + c_2 e^{2t} + c_3 e^{-2t}. \end{aligned}$$

Based on this fact, we now look for a particular solution of the form

$$y_p = Ate^{-2t}.$$

Differentiating this expression three times, one finds that

$$y_p' = Ae^{-2t}(1-2t), \qquad y_p'' = 4Ae^{-2t}(t-1), \qquad y_p''' = 4Ae^{-2t}(3-2t).$$

Once we now combine all these facts, we may finally conclude that

$$\begin{aligned} y_p''' + y_p'' - 4y_p' - 4y_p = 4Ae^{-2t} \quad &\Longrightarrow \quad A = 1 \quad \Longrightarrow \quad y_p = te^{-2t} \\ &\Longrightarrow \quad y = c_1 e^{-t} + c_2 e^{2t} + c_3 e^{-2t} + te^{-2t}. \end{aligned}$$

4a. You already did this in problem 2 of homework 4; every solution is of the form

$$x(t) = e^t(x_0 \cos t - y_0 \sin t), \qquad y(t) = e^t(x_0 \sin t + y_0 \cos t).$$

4b. According to the formula in part (a), neither x nor y remains bounded at all times, so the zero solution is unstable. Alternatively, one can compute the eigenvalues of

$$A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \quad \Longrightarrow \quad \lambda^2 - 2\lambda + 2 = 0 \quad \Longrightarrow \quad \lambda = 1 \pm i.$$

Since these eigenvalues have positive real part, the zero solution is unstable.

5a. In this case, the Jacobian matrix at the origin is

$$J = \begin{bmatrix} f_x & f_y \\ g_x & g_y \end{bmatrix} = \begin{bmatrix} 1-2x & 1-2y \\ 2-2xy & 1-x^2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}$$

so its eigenvalues are given by

$$\lambda^2 - (\operatorname{tr} J)\lambda + \det J = 0 \quad \Longrightarrow \quad \lambda^2 - 2\lambda - 1 = 0 \quad \Longrightarrow \quad \lambda = 1 \pm \sqrt{2}.$$

Since one of these eigenvalues is positive, the zero solution is unstable.

5b. Indeed, $V(x,y) = x^2 + y^2$ is positive definite with

$$V^*(x,y) = 2xx' + 2yy' = -2x^2y^2 - 2x^4 - 2y^2$$

so that $V^*(x,y) \leq 0$ at all points with equality only at the origin.

5c. In this case, $V(x,y) = x^2 + y^2$ is positive definite with

$$V^*(x,y) = 2xx' + 2yy' = -4x^2 - 2xy^2 - 2x^2y - 2y^2$$

and we need to show that $V^*(x,y) \leq 0$ in some open region around the origin. Note that the quadratic terms have the correct sign; those are also the dominant terms for points near the origin, as the remaining terms are cubic. Let us now write

$$V^*(x,y) = -2x^2(y+2) - 2y^2(x+1)$$

and focus on the region defined by $y > -2$ and $x > -1$. Then this region is open and we do have $V^*(x,y) \leq 0$ with equality only at the origin.