ODEs, Homework #5
Solutions

1. *In each of the following cases, is the zero solution stable? Asymptotically stable?*

(a) $x' = 3x - 2y,\ y' = 2x - 2y$ (b) $x' = x - 5y,\ y' = x - 3y$ (c) $x' = -y,\ y' = 4x$

- In each case, we write the given system in the form $\boldsymbol{y}' = A\boldsymbol{y}$ and then we compute the eigenvalues of A to determine stability. For the first system, we have

$$A = \begin{bmatrix} 3 & -2 \\ 2 & -2 \end{bmatrix} \quad\Longrightarrow\quad \lambda^2 - \lambda - 2 = 0 \quad\Longrightarrow\quad \lambda = -1, 2$$

and the zero solution is unstable. For the second system, we have

$$A = \begin{bmatrix} 1 & -5 \\ 1 & -3 \end{bmatrix} \quad\Longrightarrow\quad \lambda^2 + 2\lambda + 2 = 0 \quad\Longrightarrow\quad \lambda = -1 \pm i$$

and the zero solution is (asymptotically) stable. For the third system, we have

$$A = \begin{bmatrix} 0 & -1 \\ 4 & 0 \end{bmatrix} \quad\Longrightarrow\quad \lambda^2 + 4 = 0 \quad\Longrightarrow\quad \lambda = \pm 2i$$

and the zero solution is stable but not asymptotically stable.

2. *Show that the zero solution is a stable solution of the system*

$$x' = y, \qquad y' = -y - x^3$$

by finding a Lyapunov function of the form $V(x,y) = \alpha x^4 + \beta y^2$.

- We take $\alpha, \beta > 0$ so that V is positive definite and then we compute

$$V^*(x,y) = 4\alpha x^3 y + 2\beta y(-y - x^3) = (4\alpha - 2\beta)x^3 y - 2\beta y^2.$$

To ensure that this expression is non-positive, one may take $\beta = 2\alpha$ for any $\alpha > 0$.

3. *Show that the zero solution is an asymptotically stable solution of the system*

$$x' = -x - xy^2, \qquad y' = -y - x^2 y$$

by finding a strict Lyapunov function of the form $V(x,y) = \alpha x^2 + \beta y^2$.

- We take $\alpha, \beta > 0$ so that V is positive definite and then we compute

$$\begin{aligned} V^*(x,y) &= 2\alpha x(-x - xy^2) + 2\beta y(-y - x^2 y) \\ &= -2\alpha x^2 - 2\alpha x^2 y^2 - 2\beta y^2 - 2\beta x^2 y^2. \end{aligned}$$

This is certainly non-positive and vanishing only at the origin for any $\alpha, \beta > 0$.

4. *Find all critical points of the system*

$$x' = x(10 - x - y), \qquad y' = y(30 - 2x - y)$$

and then classify them as stable, asymptotically stable or unstable.

- To find the critical points, we need to solve the equations

$$x(10 - x - y) = 0, \qquad y(30 - 2x - y) = 0.$$

These imply $x = 0$ and $y(30 - y) = 0$ or else $y = 0$ and $x(10 - x) = 0$ or else

$$10 - x - y = 0, \qquad 30 - 2x - y = 0.$$

It now easily follows that there are four critical points, namely

$$A(0,0), \qquad B(0,30), \qquad C(10,0), \qquad D(20,-10).$$

To determine their stability, we compute the eigenvalues of the Jacobian matrix

$$J = \begin{bmatrix} f_x & f_y \\ g_x & g_y \end{bmatrix} = \begin{bmatrix} 10 - 2x - y & -x \\ -2y & 30 - 2x - 2y \end{bmatrix}.$$

At the point $A(0,0)$, this becomes a diagonal matrix with entries $10, 30$ so the eigenvalues are positive and $(0,0)$ is unstable. At the point $B(0,30)$, we have

$$J = \begin{bmatrix} -20 & 0 \\ -60 & -30 \end{bmatrix} \quad \Longrightarrow \quad \lambda = -20, -30$$

so this point is (asymptotically) stable. At the point $C(10,0)$, we have

$$J = \begin{bmatrix} -10 & -10 \\ 0 & 10 \end{bmatrix} \quad \Longrightarrow \quad \lambda = -10, 10$$

so this point is unstable. At the point $D(20,-10)$, we finally have

$$J = \begin{bmatrix} -20 & -20 \\ 20 & 10 \end{bmatrix} \quad \Longrightarrow \quad \lambda^2 + 10\lambda + 200 = 0 \quad \Longrightarrow \quad \lambda = -5 \pm 5i\sqrt{7}$$

so this point is (asymptotically) stable as well.

5. *Find all critical points of the system*

$$x' = y^2 - x, \qquad y' = x^2 - y$$

and then classify them as stable, asymptotically stable or unstable.

- A critical point (x, y) must satisfy the equations

$$y^2 = x, \qquad x^2 = y \quad \Longrightarrow \quad y^4 = x^2 = y \quad \Longrightarrow \quad y(y^3 - 1) = 0.$$

Thus, it easily follows that the only critical points are

$$A(0, 0), \qquad B(1, 1).$$

To determine their stability, we compute the eigenvalues of the Jacobian matrix

$$J = \begin{bmatrix} f_x & f_y \\ g_x & g_y \end{bmatrix} = \begin{bmatrix} -1 & 2y \\ 2x & -1 \end{bmatrix}.$$

At the point $A(0, 0)$, this becomes a diagonal matrix with entries $-1, -1$ so the zero solution is asymptotically stable. At the point $B(1, 1)$, we have

$$J = \begin{bmatrix} -1 & 2 \\ 2 & -1 \end{bmatrix} \quad \Longrightarrow \quad \lambda^2 + 2\lambda - 3 = 0 \quad \Longrightarrow \quad \lambda = -3, 1$$

so one of the eigenvalues is positive and this point is unstable.