ODEs, Homework #4
Solutions

1. *Check that $y_1(t) = t$ is a solution of the second-order ODE*

$$(t\cos t - \sin t)y'' + y't\sin t - y\sin t = 0$$

and then use this fact to find all solutions of the ODE.

- When $y_1 = t$, we have $y_1' = 1$ and also $y_1'' = 0$, so

$$(t\cos t - \sin t)y_1'' + y_1't\sin t - y_1\sin t = t\sin t - t\sin t = 0$$

and $y_1 = t$ is a solution, indeed. Let us now use reduction of order to find a second solution of the form $y_2 = y_1 v = tv$. Differentiating, we get

$$y_2' = v + tv', \qquad y_2'' = 2v' + tv''$$

and so $y_2 = tv$ is also a solution, provided that

$$\begin{aligned} 0 &= (t\cos t - \sin t)y_2'' + y_2't\sin t - y_2\sin t \\ &= (t\cos t - \sin t)tv'' + (2t\cos t - 2\sin t + t^2\sin t)v'. \end{aligned}$$

This gives rise to the first-order linear equation

$$v'' + \left(\frac{2}{t} + \frac{t\sin t}{t\cos t - \sin t}\right)v' = 0.$$

Noting that $(t\cos t - \sin t)' = -t\sin t$, we see that an integrating factor is

$$\begin{aligned} \mu &= \exp\left(\int \frac{2}{t} + \frac{t\sin t}{t\cos t - \sin t}\,dt\right) \\ &= \exp\Big(2\log t - \log(t\cos t - \sin t)\Big) = \frac{t^2}{t\cos t - \sin t}\,. \end{aligned}$$

Multiplying by this factor and then integrating, we conclude that

$$\begin{aligned} (\mu v')' = 0 \quad &\Longrightarrow \quad v' = \frac{C_1}{\mu} = \frac{C_1(t\cos t - \sin t)}{t^2} = \left(\frac{C_1\sin t}{t}\right)' \\ &\Longrightarrow \quad v = \frac{C_1\sin t}{t} + C_2 \\ &\Longrightarrow \quad y_2 = C_1\sin t + C_2 t. \end{aligned}$$

2. *Show that the zero solution is the only bounded solution of*

$$x'(t) = x + xy^2, \qquad y'(t) = y - x^2y.$$

- If $x(t), y(t)$ is a solution of the given system, then $E(t) = x^2 + y^2$ satisfies

$$E'(t) = 2xx' + 2yy' = 2x^2 + 2y^2 = 2E(t) \quad \Longrightarrow \quad E(t) = E(0)e^{2t}.$$

Thus, $E(t)$ can only remain bounded for all times, if it is identically zero.

3. *For which of the following ODEs is the zero solution stable? Asymptotically stable?*

(a) $x' = -2y,\ y' = 2x$ (b) $x' = x + 2y,\ y' = x$ (c) $x' = x - 5y,\ y' = 5x + y$

- In the first case, we have $\boldsymbol{y}' = A\boldsymbol{y}$, where the matrix A and its eigenvalues are

$$A = \begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix} \implies \lambda^2 + 4 = 0 \implies \lambda = \pm 2i.$$

Since those have zero real part, the zero solution is stable but not asymptotically.

- In the second case, the matrix A and its eigenvalues are

$$A = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix} \implies \lambda^2 - \lambda - 2 = 0 \implies \lambda = -1, 2.$$

Since one of the eigenvalues is positive, the zero solution is unstable.

- In the third case, finally, the matrix A and its eigenvalues are

$$A = \begin{bmatrix} 1 & -5 \\ 5 & 1 \end{bmatrix} \implies \lambda^2 - 2\lambda + 26 = 0 \implies \lambda = 1 \pm 5i.$$

Since those have positive real part, the zero solution is unstable.

4. *For which of the following ODEs is the zero solution stable? Asymptotically stable?*

(a) $x'(t) = x(t)^2$ (b) $x'(t) = -x(t)^3$ (c) $x'(t) = x(t)\cos t$

- In the first case, separation of variables gives

$$\frac{dx}{dt} = x^2 \implies -x^{-1} = t + C \implies x(t) = -\frac{1}{t + C}.$$

If we now impose the initial condition $x(0) = x_0$, then we end up with

$$x_0 = -1/C \implies x(t) = -\frac{1}{t - 1/x_0} = \frac{x_0}{1 - x_0 t}$$

and this implies blow up at time $t = 1/x_0$. Thus, the zero solution is unstable.

- In the second case, separation of variables gives

$$\frac{dx}{dt} = -x^3 \implies -\frac{1}{2}x^{-2} = -t + C \implies x(t)^2 = \frac{1}{2t + C}.$$

Imposing the initial condition $x(0) = x_0$, we then easily get

$$x_0^2 = 1/C \implies x(t)^2 = \frac{1}{2t + 1/x_0^2} = \frac{x_0^2}{2tx_0^2 + 1}.$$

In particular, $x(t) \to 0$ as $t \to \infty$ and the zero solution is asymptotically stable.

- In the last case, separation of variables gives

$$\frac{dx}{dt} = x\cos t \quad\Longrightarrow\quad \log x = \sin t + C \quad\Longrightarrow\quad x(t) = x_0 e^{\sin t},$$

so it easily follows that the zero solution is stable but not asymptotically.

5. *Use the substitution $z = \log y(t)$ to solve the equation $y' = y(\log y - 1)$.*

- Since $z = \log y$, we have $z' = \frac{1}{y}\, y'$ and so

$$z' = \frac{y'}{y} = \log y - 1 = z - 1.$$

This is a separable equation which gives

$$\begin{aligned} \frac{dz}{dt} = z - 1 \quad &\Longrightarrow \quad \log(z-1) = t + C \\ &\Longrightarrow \quad z - 1 = Ce^t \\ &\Longrightarrow \quad y = e^z = e^{Ce^t+1}. \end{aligned}$$

6. *Check that $y_1(t) = e^t$ is a solution of the second-order ODE*

$$(t^2+t)y'' - (t^2-2)y' - (t+2)y = 0$$

and then use this fact to find all solutions of the ODE.

- When $y_1 = e^t$, we have $y_1'' = y_1' = e^t$, so

$$(t^2+t)y_1'' - (t^2-2)y_1' - (t+2)y_1 = (t^2+t-t^2+2-t-2)e^t = 0$$

and $y_1 = e^t$ is a solution, indeed. Using reduction of order, we shall now find a second solution of the form $y_2 = y_1 v = e^t v$. Differentiating, we get

$$y_2 = e^t v, \qquad y_2' = e^t v + e^t v', \qquad y_2'' = e^t v + 2e^t v' + e^t v''$$

and thus $y_2 = e^t v$ is also a solution, provided that

$$0 = (t^2+t)y_2'' - (t^2-2)y_2' - (t+2)y_2 = (t^2+t)e^t v'' + (t^2+2t+2)e^t v'.$$

This gives rise to the first-order linear ODE

$$v'' + \frac{t^2+2t+2}{t^2+t}\, v' = 0$$

and the corresponding integrating factor is

$$\mu = \exp\left(\int \frac{t^2+2t+2}{t^2+t}\, dt\right) = \exp\left(\int 1 + \frac{2}{t} - \frac{1}{t+1}\, dt\right) = \frac{t^2 e^t}{t+1}.$$

Multiplying by this factor and then integrating, we conclude that

$$\begin{aligned}(\mu v')' = 0 \quad &\Longrightarrow \quad v' = \frac{C_1}{\mu} = \frac{C_1(t+1)e^{-t}}{t^2} = -\left(\frac{C_1 e^{-t}}{t}\right)' \\ &\Longrightarrow \quad v = -C_1 t^{-1} e^{-t} + C_2 \\ &\Longrightarrow \quad y_2 = -C_1 t^{-1} + C_2 e^t.\end{aligned}$$

7. *Check whether the zero solution is a stable or unstable solution of* $\boldsymbol{y}' = A\boldsymbol{y}$ *when*

$$A = \begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix}, \qquad A = \begin{bmatrix} 1 & -4 \\ 4 & -7 \end{bmatrix}, \qquad A = \begin{bmatrix} 1 & -5 \\ 1 & -3 \end{bmatrix}.$$

- In the first case, we have $\operatorname{tr} A = 0$ and $\det A = -1$, so the eigenvalues are

$$\lambda^2 - (\operatorname{tr} A)\lambda + \det A = 0 \quad \Longrightarrow \quad \lambda^2 - 1 = 0 \quad \Longrightarrow \quad \lambda = -1, 1.$$

Since one of those is positive, the zero solution is unstable.

- In the second case, we have $\operatorname{tr} A = -6$ and $\det A = 9$, so the eigenvalues are

$$\lambda^2 - (\operatorname{tr} A)\lambda + \det A = 0 \quad \Longrightarrow \quad \lambda^2 + 6\lambda + 9 = 0 \quad \Longrightarrow \quad \lambda = -3, -3.$$

Since those are both negative, the zero solution is stable.

- In the third case, we have $\operatorname{tr} A = -2$ and $\det A = 2$, so the eigenvalues are

$$\lambda^2 - (\operatorname{tr} A)\lambda + \det A = 0 \quad \Longrightarrow \quad \lambda^2 + 2\lambda + 2 = 0 \quad \Longrightarrow \quad \lambda = -1 \pm i.$$

Since those are complex with negative real part, the zero solution is stable.

8. *Use the substitution* $w = 1/y(t)$ *to solve the equation* $ty' + y = y^2 \log t$.

- Setting $y = w^{-1}$, we get $y' = -w^{-2}w'$, hence also

$$-tw^{-2}w' + w^{-1} = w^{-2}\log t \quad \Longrightarrow \quad w' - \frac{w}{t} = -\frac{\log t}{t}.$$

This is a first-order linear equation with integrating factor

$$\mu = \exp\left(-\int \frac{dt}{t}\right) = \exp(-\log t) = t^{-1}.$$

We now multiply by this factor and integrate to get

$$(t^{-1}w)' = -\frac{\log t}{t^2} \quad \Longrightarrow \quad t^{-1}w = -\int \frac{\log t}{t^2}\, dt.$$

To compute the integral, let $u = \log t$ and $dv = -t^{-2}dt$. Then $v = t^{-1}$ and so

$$-\int \frac{\log t}{t^2}\,dt = \int u\,dv = uv - \int v\,du = t^{-1}\log t - \int t^{-2}\,dt.$$

Combining the last two equations, we now get

$$t^{-1}w = t^{-1}\log t - \int t^{-2}\,dt = t^{-1}\log t + t^{-1} + C,$$

so we may finally conclude that

$$w = \log t + 1 + Ct \quad\Longrightarrow\quad y = \frac{1}{w} = \frac{1}{\log t + 1 + Ct}.$$

9. *Show that the zero solution is an unstable solution of the system*

$$x'(t) = x + 2y + xy, \qquad y'(t) = y - 2x - x^2.$$

Hint: find and solve the ODE satisfied by $E(t) = x(t)^2 + y(t)^2$.

- Following the hint, let $E(t) = x^2 + y^2$ and note that

$$E'(t) = 2xx' + 2yy' = 2x^2 + \cancel{4xy} + \cancel{2x^2y} + 2y^2 - \cancel{4xy} - \cancel{2x^2y} = 2E(t).$$

This gives a separable first-order ODE which can be easily solved to get

$$E'(t) = 2E(t) \quad\Longrightarrow\quad E(t) = Ce^{2t} \quad\Longrightarrow\quad E(t) = E(0)e^{2t}.$$

Since $E(t)$ measures distance from the origin, this means that solutions which start out near the origin do not remain near the origin, so the zero solution is unstable.

10. *Let* $a \in \mathbb{R}$ *and consider the second-order equation*

$$y''(t) + 2ay'(t) + y(t) = 0.$$

For which values of a *is the zero solution stable? Asymptotically stable?*

- In this case, the associated characteristic equation has roots

$$\lambda^2 + 2a\lambda + 1 = 0 \quad\Longrightarrow\quad \lambda = -a \pm \sqrt{a^2 - 1}.$$

These are real and negative when $a \geq 1$, but real and positive when $a \leq -1$. In the remaining case $-1 < a < 1$, the roots are complex with real part equal to $-a$. Thus, the zero solution is stable when $a \geq 0$ and asymptotically stable when $a > 0$.