ODEs, Homework #3
Solutions

1. *Suppose A, B are constant square matrices such that $e^{tA}e^{tB} = e^{t(A+B)}$ for all $t \in \mathbb{R}$. Show that $AB = BA$. Hint: differentiate twice and let $t = 0$.*

- Differentiating the given identity with respect to t, we find that

$$e^{tA}e^{tB} = e^{t(A+B)} \quad \Longrightarrow \quad Ae^{tA}e^{tB} + e^{tA}Be^{tB} = (A+B)e^{t(A+B)}.$$

Differentiating once again, we now get

$$A^2e^{tA}e^{tB} + Ae^{tA}Be^{tB} + Ae^{tA}Be^{tB} + e^{tA}B^2e^{tB} = (A+B)^2e^{t(A+B)}$$

so we can let $t = 0$ to conclude that

$$A^2 + AB + AB + B^2 = A^2 + AB + BA + B^2 \quad \Longrightarrow \quad AB = BA.$$

2. *Compute the matrix exponential e^{tA} in the case that*

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 1 & 1 & 1 \end{bmatrix}.$$

- In this case, $\lambda = 1$ is a double eigenvalue with corresponding eigenvectors

$$\boldsymbol{v}_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \qquad \boldsymbol{v}_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}.$$

There is also a simple eigenvalue, namely $\lambda = 2$, with corresponding eigenvector

$$\boldsymbol{v}_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}.$$

Since three linearly independent eigenvectors exist, A is diagonalizable and so

$$P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \quad \Longrightarrow \quad P^{-1}AP = \begin{bmatrix} 1 & & \\ & 1 & \\ & & 2 \end{bmatrix}$$

$$\Longrightarrow \quad e^{tP^{-1}AP} = \begin{bmatrix} e^t & & \\ & e^t & \\ & & e^{2t} \end{bmatrix}$$

$$\Longrightarrow \quad e^{tA} = \begin{bmatrix} e^t & 0 & 0 \\ e^{2t} - e^t & e^{2t} & 0 \\ e^{2t} - e^t & e^{2t} - e^t & e^t \end{bmatrix}.$$

3. *Compute the matrix exponential* e^{tA} *in the case that* $A = \begin{bmatrix} 1 & 2 \\ -4 & -3 \end{bmatrix}$.

- In this case, the eigenvalues of A are given by

$$\lambda^2 - (\operatorname{tr} A)\lambda + \det A = 0 \quad \Longrightarrow \quad \lambda^2 + 2\lambda + 5 = 0 \quad \Longrightarrow \quad \lambda = -1 \pm 2i.$$

Since the eigenvalues are distinct, A is diagonalizable, and it is easy to check that

$$\boldsymbol{v}_1 = \begin{bmatrix} 1 \\ i-1 \end{bmatrix}, \qquad \boldsymbol{v}_2 = \begin{bmatrix} 1 \\ -i-1 \end{bmatrix}$$

are eigenvectors corresponding to $\lambda = -1+2i$ and $\lambda = -1-2i$, respectively. Thus,

$$P = \begin{bmatrix} 1 & 1 \\ i-1 & -i-1 \end{bmatrix} \quad \Longrightarrow \quad P^{-1}AP = \begin{bmatrix} -1+2i & 0 \\ 0 & -1-2i \end{bmatrix}$$
$$\Longrightarrow \quad e^{tA} = e^{-t} \begin{bmatrix} \cos(2t)+\sin(2t) & \sin(2t) \\ -2\sin(2t) & \cos(2t)-\sin(2t) \end{bmatrix}.$$

4. *Let* $x_0, v_0 \in \mathbb{R}$ *be fixed. Find the unique solution of the initial value problem*

$$x''(t) - 2x'(t) + 2x(t) = e^t, \qquad x(0) = x_0, \qquad x'(0) = v_0.$$

- To find the homogeneous solution x_h, we note that

$$\lambda^2 - 2\lambda + 2 = 0 \quad \Longrightarrow \quad \lambda = 1 \pm i \quad \Longrightarrow \quad x_h = c_1 e^t \sin t + c_2 e^t \cos t.$$

Based on this fact, we now look for a particular solution of the form

$$x_p = Ae^t.$$

It is easy to check that this is a solution if and only if

$$e^t = x_p'' - 2x_p' + 2x_p = Ae^t \quad \Longleftrightarrow \quad A = 1.$$

Writing $x = x_h + x_p$ as usual, the initial condition $x(0) = x_0$ now gives

$$x = c_1 e^t \sin t + c_2 e^t \cos t + e^t \quad \Longrightarrow \quad x_0 = c_2 + 1 \quad \Longrightarrow \quad c_2 = x_0 - 1$$

and the initial condition $x'(0) = v_0$ gives

$$x' = c_1 e^t(\sin t + \cos t) + c_2 e^t(\cos t - \sin t) + e^t \quad \Longrightarrow \quad v_0 = c_1 + c_2 + 1.$$

In particular, $v_0 = c_1 + x_0$ and the unique solution is

$$x = (v_0 - x_0)e^t \sin t + (x_0 - 1)e^t \cos t + e^t.$$

5. *Find all solutions of the non-homogeneous scalar ODE*

$$x''(t) - 2x'(t) + 2x(t) = te^{2t}.$$

- To find the homogeneous solution x_h, we note that

$$\lambda^2 - 2\lambda + 2 = 0 \quad \Longrightarrow \quad \lambda = 1 \pm i \quad \Longrightarrow \quad x_h = c_1 e^t \sin t + c_2 e^t \cos t.$$

Based on this fact, we now look for a particular solution of the form

$$x_p = Ate^{2t} + Be^{2t}.$$

Differentiating twice, one finds that

$$\begin{aligned} x_p' &= 2Ate^{2t} + (A + 2B)e^{2t} \\ x_p'' &= 4Ate^{2t} + 4(A + B)e^{2t} \\ x_p'' - 2x_p' + 2x_p &= 2Ate^{2t} + 2(A + B)e^{2t}. \end{aligned}$$

Thus, x_p is a particular solution when $A = 1/2$ and $B = -1/2$, so

$$x = x_h + x_p = c_1 e^t \sin t + c_2 e^t \cos t + \frac{te^{2t}}{2} - \frac{e^{2t}}{2}.$$

6. *Find all solutions of the non-homogeneous third-order scalar ODE*

$$y'''(t) - 2y''(t) - y'(t) + 2y(t) = \sin t.$$

- To find the homogeneous solution y_h, we note that

$$\begin{aligned} \lambda^3 - 2\lambda^2 - \lambda + 2 = 0 \quad &\Longrightarrow \quad \lambda^2(\lambda - 2) - (\lambda - 2) = 0 \\ &\Longrightarrow \quad (\lambda - 2)(\lambda - 1)(\lambda + 1) = 0 \\ &\Longrightarrow \quad y_h = c_1 e^{2t} + c_2 e^t + c_3 e^{-t}. \end{aligned}$$

Based on this fact, we now look for a particular solution of the form

$$y_p = A \sin t + B \cos t.$$

Differentiating three times, one finds that

$$y_p' = A \cos t - B \sin t, \qquad y_p'' = -A \sin t - B \cos t, \qquad y_p''' = -A \cos t + B \sin t.$$

Once we now combine all these facts, we get

$$\sin t = y_p''' - 2y_p'' - y_p' + 2y_p = 2(2A + B) \sin t + 2(2B - A) \cos t.$$

It easily follows that $A = 1/5$ and $B = 1/10$, hence

$$y = y_h + y_p = c_1 e^{2t} + c_2 e^t + c_3 e^{-t} + \frac{\sin t}{5} + \frac{\cos t}{10}.$$

7. *Find all solutions of the non-homogeneous scalar ODE*

$$y''(t) + 2y'(t) + y(t) = 2e^{-t} + t.$$

- To find the homogeneous solution y_h, we note that

$$\lambda^2 + 2\lambda + 1 = 0 \quad \Longrightarrow \quad (\lambda+1)^2 = 0 \quad \Longrightarrow \quad y_h = c_1 e^{-t} + c_2 t e^{-t}.$$

Based on this fact, we now look for a particular solution of the form

$$y_p = At^2 e^{-t} + Bt + C.$$

Differentiating twice, one finds that

$$\begin{aligned} y_p' &= 2Ate^{-t} - At^2 e^{-t} + B \\ y_p'' &= 2Ae^{-t} - 4Ate^{-t} + At^2 e^{-t} \\ y_p'' + 2y_p' + y_p &= 2Ae^{-t} + Bt + 2B + C. \end{aligned}$$

In particular, y_p is a solution when $2A = 2$, $B = 1$ and $C = -2B$, so

$$y_p = t^2 e^{-t} + t - 2 \quad \Longrightarrow \quad y = y_h + y_p = c_1 e^{-t} + c_2 t e^{-t} + t^2 e^{-t} + t - 2.$$

8. *Find all solutions of the non-homogeneous scalar ODE*

$$y''(t) - 3y'(t) + 2y(t) = t^2 + t + 1.$$

- To find the homogeneous solution y_h, we note that

$$\lambda^2 - 3\lambda + 2 = 0 \quad \Longrightarrow \quad (\lambda-1)(\lambda-2) = 0 \quad \Longrightarrow \quad y_h = c_1 e^{t} + c_2 e^{2t}.$$

Based on this fact, we now look for a particular solution of the form

$$y_p = At^2 + Bt + C.$$

Differentiating twice, one finds that

$$\begin{aligned} y_p'' - 3y_p' + 2y_p &= 2A - 3(2At + B) + 2(At^2 + Bt + C) \\ &= 2At^2 + (2B - 6A)t + (2C - 3B + 2A). \end{aligned}$$

In particular, we need to have $2A = 2B - 6A = 2C - 3B + 2A = 1$ and so

$$A = 1/2, \qquad B = 2, \qquad C = 3 \quad \Longrightarrow \quad y = c_1 e^t + c_2 e^{2t} + \frac{t^2}{2} + 2t + 3.$$

9. *Determine the unique solution of the initial value problem*

$$x'(t) = x - y, \qquad y'(t) = x + y, \qquad x(0) = x_0, \qquad y(0) = y_0.$$

- To solve the initial value problem, we first express it as a system, namely

$$\boldsymbol{y}'(t) = A\boldsymbol{y}(t), \qquad \boldsymbol{y}(0) = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}, \qquad A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}.$$

Now, the eigenvalues of A are given by

$$\lambda^2 - (\operatorname{tr} A)\lambda + \det A = 0 \quad \Longrightarrow \quad \lambda^2 - 2\lambda + 2 = 0 \quad \Longrightarrow \quad \lambda = 1 \pm i.$$

Since the eigenvalues are distinct, A is diagonalizable, and it is easy to check that

$$\boldsymbol{v}_1 = \begin{bmatrix} 1 \\ -i \end{bmatrix}, \qquad \boldsymbol{v}_2 = \begin{bmatrix} 1 \\ i \end{bmatrix}$$

are eigenvectors corresponding to $\lambda = 1 + i$ and $\lambda = 1 - i$, respectively. Thus,

$$\begin{aligned} P = \begin{bmatrix} 1 & 1 \\ -i & i \end{bmatrix} \quad &\Longrightarrow \quad P^{-1}AP = \begin{bmatrix} 1+i & 0 \\ 0 & 1-i \end{bmatrix} \\ &\Longrightarrow \quad e^{tA} = P \cdot e^{tP^{-1}AP} \cdot P^{-1} = e^t \begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix} \end{aligned}$$

and the unique solution of the initial value problem is

$$\boldsymbol{y}(t) = e^{tA}\boldsymbol{y}(0) = e^t \begin{bmatrix} x_0 \cos t - y_0 \sin t \\ x_0 \sin t + y_0 \cos t \end{bmatrix}.$$

10. *Show that $E(t) = x(t)^2 + y(t)^2$ is decreasing for all solutions x, y of the system*

$$x'(t) = -xy^3 - x, \qquad y'(t) = x^2y^2 - y.$$

- Indeed, $E(t)$ is decreasing because

$$E'(t) = 2xx' + 2yy' = -\cancel{2x^2y^3} - 2x^2 + \cancel{2x^2y^3} - 2y^2 \leq 0.$$