ODEs, Homework #2
Solutions

1. *Find all solutions of the system of ODEs*

$$x'(t) = -12x(t) - 16y(t), \qquad y'(t) = 11x(t) + 15y(t).$$

- First of all, let us use vectors to write the given system in the form

$$\boldsymbol{y} = \begin{bmatrix} x \\ y \end{bmatrix} \quad \Longrightarrow \quad \boldsymbol{y}' = A\boldsymbol{y}, \qquad A = \begin{bmatrix} -12 & -16 \\ 11 & 15 \end{bmatrix}.$$

Since $\operatorname{tr} A = 3$ and $\det A = -4$, the eigenvalues of A are given by

$$\lambda^2 - (\operatorname{tr} A)\lambda + \det A = 0 \quad \Longrightarrow \quad \lambda^2 - 3\lambda - 4 = 0 \quad \Longrightarrow \quad \lambda = -1, 4.$$

Moreover, it is easy to check that

$$\boldsymbol{v}_1 = \begin{bmatrix} 16 \\ -11 \end{bmatrix}, \qquad \boldsymbol{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

are eigenvectors corresponding to $\lambda = -1$ and $\lambda = 4$, respectively. This implies

$$\boldsymbol{y} = c_1 e^{-t}\boldsymbol{v}_1 + c_2 e^{4t}\boldsymbol{v}_2 = \begin{bmatrix} 16c_1e^{-t} + c_2e^{4t} \\ -11c_1e^{-t} - c_2e^{4t} \end{bmatrix}.$$

2. *Let $x_0, v_0 \in \mathbb{R}$ be given. Find the unique solution of the initial value problem*

$$x''(t) - 3x'(t) + 2x(t) = 0, \qquad x(0) = x_0, \qquad x'(0) = v_0.$$

- In this case, the associated polynomial equation is

$$\lambda^2 - 3\lambda + 2 = 0 \quad \Longrightarrow \quad (\lambda - 1)(\lambda - 2) = 0 \quad \Longrightarrow \quad \lambda = 1, 2.$$

Since the two roots are distinct, the solution is given by

$$x(t) = c_1e^t + c_2e^{2t} \quad \Longrightarrow \quad x'(t) = c_1e^t + 2c_2e^{2t}.$$

To ensure that the initial conditions are satisfied, we need to ensure

$$c_1 + c_2 = x_0, \quad c_1 + 2c_2 = v_0 \quad \Longrightarrow \quad c_1 = 2x_0 - v_0, \quad c_2 = v_0 - x_0.$$

In particular, the unique solution of the initial value problem is

$$x(t) = (2x_0 - v_0)e^t + (v_0 - x_0)e^{2t}.$$

3. *The system* $\mathbf{y}' = A\mathbf{y}$ *can always be solved directly when* A *is upper triangular. Prove this in the case that* A *is* 2×2 *upper triangular with constant entries, say*

$$\begin{bmatrix} x \\ y \end{bmatrix}' = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

for some $a, b, c \in \mathbb{R}$. *Hint: solve the second equation and then solve the first.*

- The second equation is separable, so we can easily solve it to get

$$\frac{dy}{dt} = cy \quad \Longrightarrow \quad \int \frac{dy}{y} = \int c\, dt \quad \Longrightarrow \quad \log y = ct + C_0.$$

In particular, $y = C_1 e^{ct}$ and we can now turn to the first equation, namely

$$x' = ax + by \quad \Longrightarrow \quad x' - ax = by = bC_1 e^{ct}.$$

This is a first-order linear equation with integrating factor

$$\mu(t) = \exp\left(-a \int dt\right) = e^{-at}.$$

Multiplying by this factor gives a perfect derivative on the left, and we get

$$(e^{-at} x)' = bC_1 e^{ct-at}.$$

In the case that $a \neq c$, the last equation implies

$$e^{-at} x = \frac{bC_1 e^{ct-at}}{c-a} + C_2 \quad \Longrightarrow \quad x = C_3 e^{ct} + C_2 e^{at}.$$

In the case that $a = c$, on the other hand, it implies

$$e^{-at} x = bC_1 t + C_2 \quad \Longrightarrow \quad x = C_3 t e^{at} + C_2 e^{at}.$$

4. *Compute the exponential* e^{tA} *when* $A = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$.

- In this case, $\operatorname{tr} A = 4$ and $\det A = -5$, so the eigenvalues of A are given by

$$\lambda^2 - (\operatorname{tr} A)\lambda + \det A = 0 \quad \Longrightarrow \quad \lambda^2 - 4\lambda - 5 = 0 \quad \Longrightarrow \quad \lambda = -1, 5.$$

Since the eigenvalues are distinct, A is diagonalizable, and it is easy to check that

$$\boldsymbol{v}_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \qquad \boldsymbol{v}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

are eigenvectors corresponding to $\lambda = -1$ and $\lambda = 5$, respectively. In particular,

$$P = \begin{bmatrix} -1 & 1 \\ 1 & 2 \end{bmatrix} \quad \Longrightarrow \quad P^{-1}AP = \begin{bmatrix} -1 & 0 \\ 0 & 5 \end{bmatrix} \quad \Longrightarrow \quad e^{tP^{-1}AP} = \begin{bmatrix} e^{-t} & 0 \\ 0 & e^{5t} \end{bmatrix}$$

$$\Longrightarrow \quad e^{tA} = P \cdot e^{tP^{-1}AP} \cdot P^{-1} = \frac{1}{3} \begin{bmatrix} e^{5t} + 2e^{-t} & e^{5t} - e^{-t} \\ 2e^{5t} - 2e^{-t} & 2e^{5t} + e^{-t} \end{bmatrix}.$$

5. *Find all solutions $x = x(t)$ of the third-order equation $x''' - x'' - 4x' + 4x = 0$.*

- In this case, the characteristic equation gives

$$\begin{aligned} \lambda^3 - \lambda^2 - 4\lambda + 4 = 0 \quad &\Longrightarrow \quad \lambda^2(\lambda - 1) - 4(\lambda - 1) = 0 \\ &\Longrightarrow \quad (\lambda^2 - 4)(\lambda - 1) = 0 \\ &\Longrightarrow \quad x = c_1 e^{-2t} + c_2 e^{2t} + c_3 e^t. \end{aligned}$$

6. *Compute the exponential e^{tA} when $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$.*

- In this case, $\operatorname{tr} A = 5$ and $\det A = 0$, so the eigenvalues of A are given by

$$\lambda^2 - (\operatorname{tr} A)\lambda + \det A = 0 \quad \Longrightarrow \quad \lambda^2 - 5\lambda = 0 \quad \Longrightarrow \quad \lambda = 0, 5.$$

Since the eigenvalues are distinct, A is diagonalizable, and it is easy to check that

$$\boldsymbol{v}_1 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}, \qquad \boldsymbol{v}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

are eigenvectors corresponding to $\lambda = 0$ and $\lambda = 5$, respectively. In particular,

$$\begin{aligned} P = \begin{bmatrix} -2 & 1 \\ 1 & 2 \end{bmatrix} \quad &\Longrightarrow \quad P^{-1}AP = \begin{bmatrix} 0 & 0 \\ 0 & 5 \end{bmatrix} \quad \Longrightarrow \quad e^{tP^{-1}AP} = \begin{bmatrix} e^{0t} & 0 \\ 0 & e^{5t} \end{bmatrix} \\ &\Longrightarrow \quad e^{tA} = P \cdot e^{tP^{-1}AP} \cdot P^{-1} = \frac{1}{5} \begin{bmatrix} e^{5t} + 4 & 2e^{5t} - 2 \\ 2e^{5t} - 2 & 4e^{5t} + 1 \end{bmatrix}. \end{aligned}$$

7. *Compute the exponential e^{tA} when*

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{bmatrix}.$$

- The three eigenvalues are $\lambda = 1, 2, 3$ with corresponding eigenvectors

$$\boldsymbol{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \qquad \boldsymbol{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \qquad \boldsymbol{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

Letting P be the matrix whose columns are these three vectors, we now get

$$e^{tA} = P \cdot e^{tP^{-1}AP} \cdot P^{-1} = \begin{bmatrix} e^t & e^{2t} - e^t & e^{3t} - e^{2t} \\ 0 & e^{2t} & e^{3t} - e^{2t} \\ 0 & 0 & e^{3t} \end{bmatrix}.$$

8. *Find all solutions of the system of ODEs*

$$x'(t) = -3x(t) + y(t), \qquad y'(t) = -7x(t) + 5y(t).$$

- First of all, let us use vectors to write the given system in the form

$$\boldsymbol{y} = \begin{bmatrix} x \\ y \end{bmatrix} \quad \Longrightarrow \quad \boldsymbol{y}' = A\boldsymbol{y}, \qquad A = \begin{bmatrix} -3 & 1 \\ -7 & 5 \end{bmatrix}.$$

Since $\operatorname{tr} A = 2$ and $\det A = -8$, the eigenvalues of A are given by

$$\lambda^2 - (\operatorname{tr} A)\lambda + \det A = 0 \quad \Longrightarrow \quad \lambda^2 - 2\lambda - 8 = 0 \quad \Longrightarrow \quad \lambda = -2, 4.$$

Moreover, it is easy to check that

$$\boldsymbol{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \qquad \boldsymbol{v}_2 = \begin{bmatrix} 1 \\ 7 \end{bmatrix}$$

are eigenvectors corresponding to $\lambda = -2$ and $\lambda = 4$, respectively. This implies

$$\boldsymbol{y} = c_1 e^{-2t}\boldsymbol{v}_1 + c_2 e^{4t}\boldsymbol{v}_2 = \begin{bmatrix} c_1 e^{-2t} + c_2 e^{4t} \\ c_1 e^{-2t} + 7c_2 e^{4t} \end{bmatrix}.$$

9. *Show that every solution of* $x''(t) + 4x'(t) + 3x(t) = 0$ *is such that*

$$\lim_{t\to\infty} x(t) = 0.$$

- In this case, the characteristic equation gives

$$\lambda^2 + 4\lambda + 3 = 0 \quad \Longrightarrow \quad \lambda = -3, -1 \quad \Longrightarrow \quad x(t) = c_1 e^{-3t} + c_2 e^{-t}$$

for some constants c_1, c_2. It is thus easy to see that $\lim_{t\to\infty} x(t) = 0$, indeed.

10. *Determine the unique solution of the initial value problem*

$$\boldsymbol{y}'(t) = A\boldsymbol{y}(t), \qquad \boldsymbol{y}(0) = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \qquad A = \begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & 2 \\ 0 & 0 & 3 \end{bmatrix}.$$

- In this case, $\lambda = 1$ is a double eigenvalue with corresponding eigenvectors

$$\boldsymbol{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \qquad \boldsymbol{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

There is also a simple eigenvalue, namely $\lambda = 3$, with corresponding eigenvector

$$\boldsymbol{v}_3 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}.$$

Since three linearly independent eigenvectors exist, A is diagonalizable and

$$\begin{aligned} P = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \quad &\Longrightarrow \quad P^{-1}AP = \begin{bmatrix} 1 & & \\ & 1 & \\ & & 3 \end{bmatrix} \\ &\Longrightarrow \quad e^{tP^{-1}AP} = \begin{bmatrix} e^t & & \\ & e^t & \\ & & e^{3t} \end{bmatrix} \\ &\Longrightarrow \quad e^{tA} = P \cdot e^{tP^{-1}AP} \cdot P^{-1} = \begin{bmatrix} e^t & 0 & 2e^{3t} - 2e^t \\ & e^t & e^{3t} - e^t \\ & & e^{3t} \end{bmatrix}. \end{aligned}$$

In particular, the unique solution of the initial value problem is given by

$$\boldsymbol{y}(t) = e^{tA}\boldsymbol{y}(0) = \begin{bmatrix} e^t & 0 & 2e^{3t} - 2e^t \\ & e^t & e^{3t} - e^t \\ & & e^{3t} \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2e^{3t} - e^t \\ e^{3t} + e^t \\ e^{3t} \end{bmatrix}.$$