ODEs, Homework #1
Solutions

1. *Determine all functions* $y = y(t)$ *such that* $y' = \frac{2ty^2}{1+t^2}$.

- Since the given equation is separable, we may separate variables to get

$$\begin{aligned} \frac{dy}{dt} = \frac{2ty^2}{1+t^2} \quad &\Longleftrightarrow \quad \int y^{-2}\,dy = \int \frac{2t\,dt}{1+t^2} \\ &\Longleftrightarrow \quad -y^{-1} = \log(1+t^2) + C \\ &\Longleftrightarrow \quad y = -\frac{1}{\log(1+t^2)+C}\,. \end{aligned}$$

2. *Determine all functions* $y = y(x)$ *such that* $y' - \frac{y}{x+1} = x$.

- The given equation is first-order linear with integrating factor

$$\mu(x) = \exp\left(-\int \frac{dx}{x+1}\right) = \exp\bigl(-\log(x+1)\bigr) = (x+1)^{-1}.$$

Multiplying by this factor, we get a perfect derivative on the left, so

$$\begin{aligned} \left(\frac{y}{x+1}\right)' = \frac{x}{x+1} = 1 - \frac{1}{x+1} \quad &\Longrightarrow \quad \frac{y}{x+1} = x - \log(x+1) + C \\ &\Longrightarrow \quad y = (x+1)\bigl(x - \log(x+1) + C\bigr). \end{aligned}$$

3. *Which of the following ODEs are linear? Which of those are homogeneous?*

$$\begin{aligned} s^2x'(s) + s^4x(s) = s, &\qquad t^2y''(t) + y(t)y'(t) = 1, \\ xy''(x) + \sin y(x) = 0, &\qquad xy''(x) + e^xy'(x) = 0. \end{aligned}$$

- Only the first and the last ones are linear. Only the last one is homogeneous.

4. *Find all solutions of the equation*

$$y'(t) - \frac{y(t)}{t\log t} = \frac{1}{t}\,, \qquad t > 0.$$

- The given equation is first-order linear with integrating factor

$$\mu(t) = \exp\left(-\int \frac{dt}{t\log t}\right) = \exp\left(-\int \frac{du}{u}\right),$$

where $u = \log t$. This gives $\mu(t) = \exp(-\log u) = u^{-1} = (\log t)^{-1}$ and thus

$$\left(\frac{y}{\log t}\right)' = \frac{1}{t\log t} \qquad \Longrightarrow \qquad \frac{y}{\log t} = \int \frac{dt}{t\log t} = \int \frac{du}{u}$$

with $u = \log t$ as before. Evaluating the integral and simplifying, we now get

$$\frac{y}{\log t} = \log u + C = \log(\log t) + C \qquad \Longrightarrow \qquad y = \big(\log(\log t) + C\big)\log t.$$

5. *Find the unique solution $y = y(t)$ of the initial value problem*

$$y' = y(1-y)\cos t, \qquad y(0) = y_0.$$

Hint: separate variables and then use partial fractions.

- First of all, let us separate variables to get

$$\frac{dy}{dt} = y(1-y)\cos t \qquad \Longrightarrow \qquad \int \frac{dy}{y(1-y)} = \int \cos t\, dt.$$

To compute the integral on the left, one has to use partial fractions to find that

$$\frac{1}{y(1-y)} = \frac{1}{y} + \frac{1}{1-y}.$$

Using this fact and a little bit of algebra, we now get

$$\log y - \log(1-y) = \sin t + C \qquad \Longrightarrow \qquad \log\frac{y}{1-y} = \sin t + C$$
$$\Longrightarrow \qquad \frac{y}{1-y} = e^{\sin t + C} = Ce^{\sin t}.$$

Due to the initial condition $y(0) = y_0$, we must have $\frac{y_0}{1-y_0} = C$ and this gives

$$\frac{y}{1-y} = \frac{y_0 e^{\sin t}}{1-y_0} \qquad \Longrightarrow \qquad y(1-y_0) = y_0 e^{\sin t} - y y_0 e^{\sin t}$$
$$\Longrightarrow \qquad y = \frac{y_0 e^{\sin t}}{1 - y_0 + y_0 e^{\sin t}}.$$

6. *Let $a, y_0 \in \mathbb{R}$ and suppose that f is continuous. Solve the initial value problem*

$$y'(t) - ay(t) = f(t), \qquad y(0) = y_0.$$

- The given equation is first-order linear with integrating factor

$$\mu(t) = \exp\left(-\int a\, dt\right) = e^{-at}.$$

We multiply the ODE by this factor and then we integrate to get

$$(ye^{-at})' = e^{-at} f(t) \quad \Longrightarrow \quad \Big[y(s)e^{-as}\Big]_0^t = \int_0^t e^{-as} f(s)\, ds.$$

Next, we simplify the last equation and we solve for y; this gives

$$y(t)e^{-at} - y_0 = \int_0^t e^{-as} f(s)\, ds \quad \Longrightarrow \quad y(t) = y_0 e^{at} + e^{at} \int_0^t e^{-as} f(s)\, ds.$$

7. *Show that the initial value problem*

$$ty'(t) = y(t), \qquad y(0) = 1$$

has no solutions. Why doesn't this contradict our existence result, Theorem 3.1?

- When $t = 0$, the ODE gives $y(0) = 0$ and this is contrary to the initial condition. The fact that no solution exists does not contradict Theorem 3.1 because $f(t, y) = y/t$ is not continuous in any rectangle around the point $(0, 1)$.

8. *Solve the initial value problem*

$$y'(t) - \frac{2y(t)}{t} = 4t^3, \qquad y(1) = 3.$$

- The given equation is first-order linear with integrating factor

$$\mu(t) = \exp\left(-\int \frac{2}{t}\, dt\right) = \exp\Big(-2\log t\Big) = t^{-2}.$$

We multiply the ODE by this factor and then we integrate to get

$$(t^{-2}y)' = 4t \quad \Longrightarrow \quad t^{-2}y = 2t^2 + C \quad \Longrightarrow \quad y = 2t^4 + Ct^2.$$

To ensure that $y(1) = 3$, we need to have $C = 1$, and this implies $y = 2t^4 + t^2$.

9. *Find all solutions of the equation*

$$(x-2)y'(x) = (x-1)y(x).$$

- Separating variables and integrating, one finds that

$$(x-2)\frac{dy}{dx} = (x-1)y \quad \Longrightarrow \quad \int \frac{dy}{y} = \int \frac{x-1}{x-2}\, dx = \int 1 + \frac{1}{x-2}\, dx.$$

Once we now compute these two integrals and simplify, we arrive at

$$\log y = x + \log(x-2) + C \quad \Longrightarrow \quad y = C(x-2)e^x.$$

10. *Show that the unique solution of the initial value problem*

$$y'(t) = \frac{2t}{1+t^2} \cdot \sin y(t), \qquad y(0) = 1$$

exists for all times. Hint: use the associated integral equation.

- The unique local solution satisfies the associated integral equation

$$y(t) = 1 + \int_0^t \frac{2s}{1+s^2} \cdot \sin y(s)\, ds.$$

Using the fact that $|\sin x| \leq 1$ for all x, we then easily find that

$$|y(t)| \leq 1 + \int_0^t \frac{2s}{1+s^2}\, ds = 1 + \log(1+t^2).$$

Thus, $y(t)$ is finite whenever t is finite and the solution exists for all times.