9 Method of undetermined coefficients

Theorem 9.1 (Undetermined coefficients). Consider the non-homogeneous ODE

$$y^{(n)} + a_{n-1}y^{(n-1)} + \ldots + a_1 y' + a_0 y = f(t).$$

If $f(t)$ is a polynomial of degree m, then this ODE has a solution of the form

$$y_p = t^k(b_0 + b_1 t + \ldots + b_m t^m).$$

If $f(t)$ is $e^{\beta t}$ times a polynomial of degree m, then it has a solution of the form

$$y_p = t^k e^{\beta t}(b_0 + b_1 t + \ldots + b_m t^m).$$

If either $f(t) = \sin(\beta t)$ or $f(t) = \cos(\beta t)$, then it has a solution of the form

$$y_p = t^k \Big(A\sin(\beta t) + B\cos(\beta t) \Big).$$

In each of these cases, $k \geq 0$ denotes the least integer such that y_p does not contain terms which already appear in the solution y_h to the associated homogeneous ODE.

Example 9.2. Consider the non-homogeneous ODE

$$y'' - 4y' + 4y = t + e^{2t} + \cos(3t).$$

In this case, the homogeneous solution y_h is given by

$$\lambda^2 - 4\lambda + 4 = 0 \quad \Longrightarrow \quad (\lambda - 2)^2 = 0 \quad \Longrightarrow \quad y_h = c_1 e^{2t} + c_2 t e^{2t}.$$

According to the theorem above, the non-homogeneous ODE has a solution of the form

$$y_p = At + B + Ct^2 e^{2t} + D\sin(3t) + E\cos(3t).$$

Example 9.3. Consider the non-homogeneous ODE

$$y'' - 3y' + 2y = t^2 + e^{2t} + te^{3t}.$$

In this case, the homogeneous solution y_h is given by

$$\lambda^2 - 3\lambda + 2 = 0 \quad \Longrightarrow \quad (\lambda - 1)(\lambda - 2) = 0 \quad \Longrightarrow \quad y_h = c_1 e^t + c_2 e^{2t}.$$

According to the theorem above, the non-homogeneous ODE has a solution of the form

$$y_p = At^2 + Bt + C + Dte^{2t} + (Et + F)e^{3t}.$$