8 Homogeneous with constant coefficients

Theorem 8.1 (Scalar homogeneous). Consider the nth-order scalar ODE

$$y^{(n)} + a_{n-1}y^{(n-1)} + \ldots + a_1 y' + a_0 y = 0$$

together with the associated characteristic equation

$$\lambda^n + a_{n-1}\lambda^{n-1} + \ldots + a_1\lambda + a_0 = 0.$$

If $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$ are the complex roots of this equation, then the solution

$$y(t) = c_1 y_1(t) + \ldots + c_n y_n(t)$$

of the ODE can be determined by associating each root λ_k with a function y_k as follows.

- A real root λ is associated with $e^{\lambda t}$, if it is a simple root. If it has multiplicity $m \geq 2$, then it is associated with $e^{\lambda t}, te^{\lambda t}, \ldots, t^{m-1}e^{\lambda t}$.
- The complex roots $\lambda = \alpha \pm i\beta$ are associated with $e^{\alpha t}\sin(\beta t)$ and $e^{\alpha t}\cos(\beta t)$, if they are simple roots. If they have multiplicity $m \geq 2$, then they are associated with

$$e^{\alpha t}\sin(\beta t), e^{\alpha t}\cos(\beta t), \ldots, t^{m-1}e^{\alpha t}\sin(\beta t), t^{m-1}e^{\alpha t}\cos(\beta t).$$

Example 8.2 (Double root). In the case that $y'' - 4y' + 4y = 0$, we have

$$\lambda^2 - 4\lambda + 4 = 0 \quad \Longrightarrow \quad (\lambda - 2)^2 = 0 \quad \Longrightarrow \quad y = c_1 e^{2t} + c_2 t e^{2t}.$$

Example 8.3 (Complex roots). In the case that $y'' - 4y' + 13y = 0$, we have

$$\lambda^2 - 4\lambda + 13 = 0 \quad \Longrightarrow \quad \lambda = 2 \pm 3i \quad \Longrightarrow \quad y = c_1 e^{2t}\sin(3t) + c_2 e^{2t}\cos(3t).$$

Example 8.4 (Third-order). In the case that $y''' - 2y'' - 4y' + 8y = 0$, we have

$$\begin{aligned} \lambda^3 - 2\lambda^2 - 4\lambda + 8 = 0 \quad &\Longrightarrow \quad \lambda^2(\lambda - 2) - 4(\lambda - 2) = 0 \\ &\Longrightarrow \quad (\lambda - 2)(\lambda - 2)(\lambda + 2) = 0 \\ &\Longrightarrow \quad y = c_1 e^{2t} + c_2 t e^{2t} + c_3 e^{-2t}. \end{aligned}$$

Example 8.5 (Fourth-order). In the case that $y'''' - y = 0$, we have

$$\begin{aligned} \lambda^4 - 1 = 0 \quad &\Longrightarrow \quad (\lambda^2 + 1)(\lambda^2 - 1) = 0 \\ &\Longrightarrow \quad \lambda = \pm i, \quad \lambda = \pm 1 \\ &\Longrightarrow \quad y = c_1 \sin t + c_2 \cos t + c_3 e^t + c_4 e^{-t}. \end{aligned}$$