7 Exponentials: Some examples

Example 7.1 (2 × 2 diagonalizable). We compute e^{tA} in the case that

$$A = \begin{bmatrix} 1 & 2 \\ 5 & 4 \end{bmatrix}.$$

First of all, let us determine the eigenvalues of A by solving the equation

$$\lambda^2 - (\operatorname{tr} A)\lambda + \det A = 0 \quad \Longrightarrow \quad \lambda^2 - 5\lambda - 6 = 0 \quad \Longrightarrow \quad \lambda = -1, 6.$$

Since the eigenvalues are distinct, A is diagonalizable, and it is easy to check that

$$\boldsymbol{v}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \qquad \boldsymbol{v}_2 = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$$

are eigenvectors corresponding to $\lambda = -1$ and $\lambda = 6$, respectively. This gives

$$P = \begin{bmatrix} 1 & 2 \\ -1 & 5 \end{bmatrix} \quad \Longrightarrow \quad P^{-1}AP = \begin{bmatrix} -1 & 0 \\ 0 & 6 \end{bmatrix} \quad \Longrightarrow \quad e^{tP^{-1}AP} = \begin{bmatrix} e^{-t} & 0 \\ 0 & e^{6t} \end{bmatrix}$$

and we can now use formula (6.1) to conclude that

$$\begin{aligned} e^{tA} = P \cdot e^{tP^{-1}AP} \cdot P^{-1} &= \begin{bmatrix} 1 & 2 \\ -1 & 5 \end{bmatrix} \begin{bmatrix} e^{-t} & 0 \\ 0 & e^{6t} \end{bmatrix} \begin{bmatrix} 5/7 & -2/7 \\ 1/7 & 1/7 \end{bmatrix} \\ &= \frac{1}{7} \begin{bmatrix} 2e^{6t} + 5e^{-t} & 2e^{6t} - 2e^{-t} \\ 5e^{6t} - 5e^{-t} & 5e^{6t} + 2e^{-t} \end{bmatrix}. \end{aligned}$$

Example 7.2 (2 × 2 with one Jordan block). We compute e^{tA} in the case that

$$A = \begin{bmatrix} 4 & 1 \\ -4 & 0 \end{bmatrix}.$$

In this case, the eigenvalues are given by

$$\lambda^2 - (\operatorname{tr} A)\lambda + \det A = 0 \quad \Longrightarrow \quad \lambda^2 - 4\lambda + 4 = 0 \quad \Longrightarrow \quad \lambda = 2$$

and it is easy to check that all eigenvectors are nonzero scalar multiples of

$$\boldsymbol{v}_1 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}.$$

This means that we are missing an eigenvector and that the Jordan form is

$$P^{-1}AP = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix} \quad \Longrightarrow \quad e^{tP^{-1}AP} = \begin{bmatrix} e^{2t} & te^{2t} \\ 0 & e^{2t} \end{bmatrix}.$$

To actually find the columns of P, we need to find vectors $\boldsymbol{v}_1, \boldsymbol{v}_2$ such that

$$A\boldsymbol{v}_1 = 2\boldsymbol{v}_1, \qquad A\boldsymbol{v}_2 = \boldsymbol{v}_1 + 2\boldsymbol{v}_2.$$

Thus, we can take $\boldsymbol{v}_1$ as the eigenvector above, and we can take $\boldsymbol{v}_2$ so that

$$(A-2I)\boldsymbol{v}_2 = \boldsymbol{v}_1 \quad\Longrightarrow\quad \boldsymbol{v}_2 = \begin{bmatrix} a \\ 1-2a \end{bmatrix} \quad\Longrightarrow\quad \boldsymbol{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

for instance. Then these two vectors give the columns of P and we get

$$e^{tA} = P\cdot e^{tP^{-1}AP}\cdot P^{-1} = e^{2t}\begin{bmatrix} 1+2t & t \\ -4t & 1-2t \end{bmatrix}.$$

Example 7.3 (3 × 3 with one Jordan block). We compute e^{tA} in the case that

$$A = \begin{bmatrix} 3 & 1 & 2 \\ 0 & 3 & 4 \\ 0 & 0 & 3 \end{bmatrix}.$$

Here, the only eigenvalue is $\lambda = 3$ and all eigenvectors are nonzero scalar multiples of

$$\boldsymbol{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

It is easy to find vectors $\boldsymbol{v}_2, \boldsymbol{v}_3$ such that $(A-3I)\boldsymbol{v}_2 = \boldsymbol{v}_1$ and $(A-3I)\boldsymbol{v}_3 = \boldsymbol{v}_2$, say

$$\boldsymbol{v}_2 = \begin{bmatrix} a \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \qquad \boldsymbol{v}_3 = \begin{bmatrix} a \\ -1/2 \\ 1/4 \end{bmatrix} = \begin{bmatrix} 0 \\ -1/2 \\ 1/4 \end{bmatrix}.$$

Then these three vectors give the columns of P and we get

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1/2 \\ 0 & 0 & 1/4 \end{bmatrix} \quad\Longrightarrow\quad P^{-1}AP = \begin{bmatrix} 3 & 1 & 0 \\ & 3 & 1 \\ & & 3 \end{bmatrix} \quad\Longrightarrow\quad e^{tP^{-1}AP} = e^{3t}\begin{bmatrix} 1 & t & \frac{t^2}{2!} \\ & 1 & t \\ & & 1 \end{bmatrix}.$$

Using this fact together with formula (6.1), it is now easy to check that

$$e^{tA} = P\cdot e^{tP^{-1}AP}\cdot P^{-1} = e^{3t}\begin{bmatrix} 1 & t & 2t^2+2t \\ & 1 & 4t \\ & & 1 \end{bmatrix}.$$

Example 7.4 (3 × 3 with two Jordan blocks). We compute e^{tA} in the case that

$$A = \begin{bmatrix} 3 & 3 & 3 \\ 0 & 3 & 3 \\ 0 & 0 & 2 \end{bmatrix}.$$

In this case, $\lambda = 2$ is a simple eigenvalue with corresponding eigenvector

$$\boldsymbol{v}_3 = \begin{bmatrix} 6 \\ -3 \\ 1 \end{bmatrix}.$$

There is also a double eigenvalue, namely $\lambda = 3$, with only one eigenvector

$$\boldsymbol{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

so we need to find a vector $\boldsymbol{v}_2$ such that $(A - 3I)\boldsymbol{v}_2 = \boldsymbol{v}_1$, say

$$\boldsymbol{v}_2 = \begin{bmatrix} a \\ 1/3 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1/3 \\ 0 \end{bmatrix}.$$

Then these three vectors give the columns of P and we get

$$P = \begin{bmatrix} 1 & 0 & 6 \\ 0 & 1/3 & -3 \\ 0 & 0 & 1 \end{bmatrix} \quad \Longrightarrow \quad P^{-1}AP = \begin{bmatrix} 3 & 1 & \\ & 3 & \\ & & 2 \end{bmatrix} \quad \Longrightarrow \quad e^{tP^{-1}AP} = \begin{bmatrix} e^{3t} & te^{3t} & \\ & e^{3t} & \\ & & e^{2t} \end{bmatrix}.$$

Combining this fact with formula (6.1), we may thus conclude that

$$e^{tA} = P \cdot e^{tP^{-1}AP} \cdot P^{-1} = \begin{bmatrix} e^{3t} & 3te^{3t} & (9t-6)e^{3t} + 6e^{2t} \\ & e^{3t} & 3e^{3t} - 3e^{2t} \\ & & e^{2t} \end{bmatrix}.$$

Example 7.5 (2×2 with complex eigenvalues). We compute e^{tA} in the case that

$$A = \begin{bmatrix} 5 & 3 \\ -6 & -1 \end{bmatrix}.$$

In this case, we have $\operatorname{tr} A = 4$ and $\det A = 13$, so the eigenvalues are given by

$$\lambda^2 - (\operatorname{tr} A)\lambda + \det A = 0 \quad \Longrightarrow \quad \lambda^2 - 4\lambda + 13 = 0 \quad \Longrightarrow \quad \lambda = 2 \pm 3i.$$

Since the eigenvalues are distinct, A is diagonalizable, and it is easy to check that

$$\boldsymbol{v}_1 = \begin{bmatrix} 1 \\ i-1 \end{bmatrix}, \qquad \boldsymbol{v}_2 = \begin{bmatrix} 1 \\ -i-1 \end{bmatrix}$$

are eigenvectors corresponding to $\lambda = 2 + 3i$ and $\lambda = 2 - 3i$, respectively. This gives

$$P = \begin{bmatrix} 1 & 1 \\ i-1 & -i-1 \end{bmatrix} \quad \Longrightarrow \quad P^{-1}AP = \begin{bmatrix} 2+3i & \\ & 2-3i \end{bmatrix}$$

and we can now use Euler's formula $e^{i\theta} = \cos\theta + i\sin\theta$ to get

$$e^{tP^{-1}AP} = e^{2t} \begin{bmatrix} \cos(3t) + i\sin(3t) & \\ & \cos(3t) - i\sin(3t) \end{bmatrix}.$$

Using formula (6.1) as before, we may finally deduce that

$$e^{tA} = P \cdot e^{tP^{-1}AP} \cdot P^{-1} = e^{2t} \begin{bmatrix} \cos(3t) + \sin(3t) & \sin(3t) \\ -2\sin(3t) & \cos(3t) - \sin(3t) \end{bmatrix}.$$