6 Exponential of a matrix

Definition 6.1. Given a square matrix A, we define its exponential e^A as the series

$$e^A = I + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \ldots = \sum_{k=0}^{\infty} \frac{A^k}{k!}.$$

It is easy to check that this series converges for any square matrix A whatsoever.

Theorem 6.2 (Constant matrices). If A is a constant square matrix, then $\Phi(t) = e^{At}$ is a fundamental matrix for the system $\boldsymbol{y}'(t) = A\boldsymbol{y}$. Moreover, the initial value problem

$$\boldsymbol{y}'(t) = A\boldsymbol{y}, \qquad \boldsymbol{y}(0) = \boldsymbol{y}_0$$

has a unique solution which is defined for all times, namely $\boldsymbol{y}(t) = e^{At}\boldsymbol{y}_0$.

Example 6.3 (Computation of e^{At}). When A is a diagonal matrix, we have

$$A = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} \quad \Longrightarrow \quad e^{At} = \begin{bmatrix} e^{\lambda_1 t} & & \\ & \ddots & \\ & & e^{\lambda_n t} \end{bmatrix}.$$

When A is a 2×2 Jordan block, we have

$$A = \begin{bmatrix} \lambda & 1 \\ & \lambda \end{bmatrix} \quad \Longrightarrow \quad e^{At} = \begin{bmatrix} e^{\lambda t} & te^{\lambda t} \\ & e^{\lambda t} \end{bmatrix}.$$

When A is a 3×3 Jordan block, we have

$$A = \begin{bmatrix} \lambda & 1 & \\ & \lambda & 1 \\ & & \lambda \end{bmatrix} \quad \Longrightarrow \quad e^{At} = \begin{bmatrix} e^{\lambda t} & te^{\lambda t} & \frac{t^2}{2!} e^{\lambda t} \\ & e^{\lambda t} & te^{\lambda t} \\ & & e^{\lambda t} \end{bmatrix}.$$

Using these facts, one can compute the exponential of any square matrix A. Namely, one may determine the Jordan form $P^{-1}AP$ and then use the formula

$$e^{At} = P \cdot e^{tP^{-1}AP} \cdot P^{-1} \tag{6.1}$$

to relate the exponential of the given matrix A to that of its Jordan form.

Lemma 6.4 (Product rule). If A, B are square matrices, then $(AB)' = A'B + AB'$.

Lemma 6.5. If A, B are square matrices that commute, then $e^{A+B} = e^A e^B$.