5 Linear homogeneous systems

Definition 5.1 (Linear dependence). The vector-valued functions $\boldsymbol{y}_1, \ldots, \boldsymbol{y}_n$ are said to be linearly dependent, if there exist constants $c_1, \ldots, c_n$ that are not all zero with

$$c_1\boldsymbol{y}_1 + \ldots + c_n\boldsymbol{y}_n = \mathbf{0} \quad \text{for all } t.$$

Theorem 5.2. Suppose $A(t)$ is an $n \times n$ matrix whose entries are continuous for all t. Then the solutions $\boldsymbol{y} \in \mathbb{R}^n$ of the linear homogeneous system

$$\boldsymbol{y}'(t) = A(t)\boldsymbol{y}(t) \tag{5.1}$$

form an n-dimensional vector space over the real numbers.

Theorem 5.3. Suppose that $\boldsymbol{y}_1, \boldsymbol{y}_2, \ldots, \boldsymbol{y}_n$ are solutions of (5.1). Then their Wronskian

$$W(t) = \det \begin{bmatrix} | & | & & | \\ \boldsymbol{y}_1 & \boldsymbol{y}_2 & \cdots & \boldsymbol{y}_n \\ | & | & & | \end{bmatrix}$$

is either zero at all points (and the $\boldsymbol{y}_i$'s are linearly dependent) or else nonzero at all points (and the $\boldsymbol{y}_i$'s are linearly independent).

Example 5.4 (Constant matrices). Consider the system $\boldsymbol{y}'(t) = A\boldsymbol{y}$ in the case that A is a constant matrix with n linearly independent eigenvectors $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n$ and corresponding real eigenvalues $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$. Then it is easy to check that $e^{\lambda_1 t}\boldsymbol{v}_1, \ldots, e^{\lambda_n t}\boldsymbol{v}_n$ are linearly independent solutions to the system, hence every solution is a linear combination of those.

Definition 5.5 (Fundamental matrix). An $n \times n$ matrix $\Phi(t)$ is a fundamental matrix for the system (5.1), if the columns of $\Phi(t)$ are linearly independent solutions. In that case, every solution to the system has the form $y(t) = \Phi(t)C$ for some constant vector C.

Example 5.6 (Higher-order scalar ODE). Consider the nth-order scalar ODE

$$y^{(n)} + a_{n-1}y^{(n-1)} + \ldots + a_1 y' + a_0 y = 0$$

in the case that the corresponding polynomial equation

$$\lambda^n + a_{n-1}\lambda^{n-1} + \ldots + a_1\lambda + a_0 = 0$$

has n distinct real roots $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$. Then every solution of the ODE has the form

$$y(t) = c_1 e^{\lambda_1 t} + \ldots + c_n e^{\lambda_n t}.$$