4 First-order systems

Remark 4.1. Every ODE and every system of ODEs can be reduced to the form

$$\boldsymbol{y}'(t) = \boldsymbol{f}(t, \boldsymbol{y})$$

for some vector $\boldsymbol{y}$ and a vector-valued function $\boldsymbol{f}$. Consider the third-order ODE

$$x'''(t) - 2x''(t) + tx(t)^2 = 0,$$

for instance. Introducing a vector whose components are x, x' and x'', we find that

$$\boldsymbol{y} = \begin{bmatrix} x \\ x' \\ x'' \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \implies \boldsymbol{y}' = \begin{bmatrix} x' \\ x'' \\ 2x'' - tx^2 \end{bmatrix} = \begin{bmatrix} y_2 \\ y_3 \\ 2y_3 - ty_1^2 \end{bmatrix} = \boldsymbol{f}(t, \boldsymbol{y}).$$

Definition 4.2 (Norm). Given a matrix A, we define its norm $|A|$ by

$$|A| = \sum_{i,j} |a_{ij}|.$$

Using this definition, one can easily check that $|A + B| \leq |A| + |B|$ and $|AB| \leq |A|\,|B|$.

Theorem 4.3 (Existence of solutions). Suppose that $\boldsymbol{f}, \boldsymbol{f}_{y_1}, \ldots, \boldsymbol{f}_{y_n}$ are continuous in an arbitrary set $A \subset \mathbb{R}^{n+1}$ that contains the point $(t_0, \boldsymbol{y}_0)$. Then the initial value problem

$$\boldsymbol{y}'(t) = \boldsymbol{f}(t, \boldsymbol{y}(t)), \qquad \boldsymbol{y}(t_0) = \boldsymbol{y}_0$$

has a unique solution that can be extended for all times, unless either its graph meets the boundary of A or else $|\boldsymbol{y}(t)|$ blows up in finite time.