3 Existence and uniqueness

Theorem 3.1 (Local existence). Suppose that f, f_y are continuous in a rectangle

$$R = \left\{(t,y) \in \mathbb{R}^2 \; : \; |t - t_0| \leq a, \quad |y - y_0| \leq b\right\}$$

which is centered around the point (t_0, y_0). Then the initial value problem

$$y'(t) = f(t, y(t)), \qquad y(t_0) = y_0 \tag{3.1}$$

has a unique solution which is defined when $|t - t_0| \leq \varepsilon$ for some $\varepsilon > 0$.

Lemma 3.2 (Integral equation). A function $y = y(t)$ is a differentiable solution of the initial value problem (3.1) if and only if y is a continuous solution of the integral equation

$$y(t) = y_0 + \int_{t_0}^{t} f(s, y(s))\, ds. \tag{3.2}$$

Theorem 3.3 (Continuation of solutions). Suppose f, f_y are continuous in an arbitrary set $A \subset \mathbb{R}^2$ that contains (t_0, y_0). Then the unique solution of the initial value problem

$$y'(t) = f(t, y(t)), \qquad y(t_0) = y_0 \tag{3.3}$$

can be extended for all times, unless either its graph meets the boundary of A or else $y(t)$ becomes infinite in finite time (in which case we say that it blows up in finite time).

Theorem 3.4 (Gronwall inequality). Suppose f, g are non-negative and continuous with

$$f(t) \leq C + \int_{t_0}^{t} f(s) g(s)\, ds \tag{3.4}$$

for some fixed constants t_0 and C. Then it must be the case that

$$f(t) \leq C \exp\left(\int_{t_0}^{t} g(s)\, ds\right).$$