2010 final exam

1a. In this case, the characteristic equation gives

$$\lambda^2 + 1 = 0 \quad \Longrightarrow \quad \lambda = \pm i \quad \Longrightarrow \quad x(t) = c_1 \sin t + c_2 \cos t$$

for some constants c_1, c_2. In particular, we have $|x(t)| \leq |c_1| + |c_2|$.

1b. The solution of this non-homogeneous equation has the form

$$x(t) = c_1 \sin t + c_2 \cos t + At \sin t + Bt \cos t$$

for some arbitrary constants c_1, c_2 and some constants A, B that one can determine, if needed. Since $x(t)$ is not a solution of the homogeneous equation, one of A, B must be nonzero, and this already implies that $x(t)$ is unbounded.

1c. In this case, the method of undetermined coefficients gives

$$x(t) = c_1 \sin t + c_2 \cos t + A \sin(2t) + B \cos(2t),$$

so the solution is easily seen to be bounded.

1d. Indeed, the ODE implies $x(0) = 0$ and this is contrary to the initial condition.

1e. Using separation of variables to solve the ODE, we get

$$t \, \frac{dx}{dt} = x \quad \Longrightarrow \quad \int \frac{dx}{x} = \int \frac{dt}{t} \quad \Longrightarrow \quad \log x = \log t + C.$$

Thus, $x = Ct$ and the initial condition $x(0) = 0$ is satisfied for any C whatsoever.

2a. The given ODE is first-order linear with integrating factor

$$\mu = \exp\left(-\int \frac{t\,dt}{t^2 + 1} \right) = \exp\left(-\frac{1}{2} \log(t^2 + 1) \right) = (t^2 + 1)^{-1/2}.$$

Multiplying by this factor and integrating, we now get

$$(\mu y)' = \frac{t}{\sqrt{t^2 + 1}} \quad \Longrightarrow \quad \mu y = \int \frac{2t\,dt}{2\sqrt{t^2 + 1}} = \int \frac{du}{2\sqrt{u}} = \sqrt{u} + C$$

using the substitution $u = t^2 + 1$. This also implies that

$$\frac{y}{\sqrt{t^2 + 1}} = \sqrt{t^2 + 1} + C \quad \Longrightarrow \quad y = t^2 + 1 + C\sqrt{t^2 + 1}.$$

Since we need to have $y(0) = 0$, it easily follows that $y = t^2 + 1 - \sqrt{t^2 + 1}$.

2b. Since $z = y^{-1}$, we have $z' = -y^{-2}y'$, hence

$$z' - Pz = -Q \quad \Longleftrightarrow \quad -y^{-2}y' - Py^{-1} = -Q \quad \Longleftrightarrow \quad y' + Py = Qy^2.$$

2c. Setting $z = y^{-1}$ as in the previous part, we end up with the ODE

$$z' - \frac{z}{t} = -\frac{\log t}{t},$$

which is first-order linear with integrating factor

$$\mu = \exp\left(-\int \frac{dt}{t}\right) = \exp(-\log t) = t^{-1}.$$

We now multiply by this factor and integrate to get

$$(t^{-1}z)' = -\frac{\log t}{t^2} \quad \Longrightarrow \quad t^{-1}z = -\int \frac{\log t}{t^2}\, dt.$$

To compute the integral, let $u = \log t$ and $dv = t^{-2}dt$. Then $v = -t^{-1}$ and so

$$\int \frac{\log t}{t^2}\, dt = \int u\, dv = uv - \int v\, du = -t^{-1}\log t + \int t^{-2}\, dt.$$

Combining the last two equations, we now get

$$t^{-1}z = t^{-1}\log t - \int t^{-2}\, dt = t^{-1}\log t + t^{-1} + C,$$

so we may finally conclude that

$$z = \log t + 1 + Ct \quad \Longrightarrow \quad y = z^{-1} = (\log t + 1 + Ct)^{-1}.$$

3a. The fact that $y_1 = e^t$ is a solution follows by the computation

$$(t+1)y_1'' - (t+2)y_1' + y_1 = (t + 1 - t - 2 + 1)e^t = 0.$$

We now use reduction of order to find a second solution of the form $y_2 = e^t v$. Since

$$y_2 = e^t v, \qquad y_2' = e^t(v + v'), \qquad y_2'' = e^t(v + 2v' + v''),$$

we see that $y_2 = e^t v$ is also a solution, provided that

$$0 = (t+1)y_2'' - (t+2)y_2' + y_2 = (t+1)e^t v'' + te^t v'.$$

Dividing through by $(t+1)e^t$ gives a linear ODE with integrating factor

$$\mu = \exp\left(\int \frac{t}{t+1}\, dt\right) = \exp\left(\int 1 - \frac{1}{t+1}\, dt\right) = \frac{e^t}{t+1}.$$

We now multiply by this factor and we integrate to get

$$(\mu v')' = 0 \quad \Longrightarrow \quad v' = C_1/\mu = C_1 e^{-t}(t+1).$$

Using this fact and an integration by parts, we conclude that

$$v = C_1 \int e^{-t}(t+1)\,dt = -C_1(t+1)e^{-t} - C_1 e^{-t} + C_2,$$

hence

$$y_2 = e^t v = -C_1(t+1) - C_1 + C_2 e^t = -C_1(t+2) + C_2 e^t.$$

3b. To find the homogeneous solution y_h, we note that

$$\begin{aligned} \lambda^3 - \lambda^2 - 4\lambda + 4 = 0 \quad &\Longrightarrow \quad \lambda^2(\lambda-1) - 4(\lambda-1) = 0 \\ &\Longrightarrow \quad (\lambda-1)(\lambda-2)(\lambda+2) = 0 \\ &\Longrightarrow \quad y_h = c_1 e^t + c_2 e^{2t} + c_3 e^{-2t}. \end{aligned}$$

Based on this fact, we now look for a particular solution of the form

$$y_p = Ate^t.$$

Differentiating three times, one finds that

$$y_p' = Ae^t(t+1), \qquad y_p'' = Ae^t(t+2), \qquad y_p''' = Ae^t(t+3).$$

In particular, $y_p''' - y_p'' - 4y_p' + 4y_p = -3Ae^t$ and this implies

$$A = -1 \quad \Longrightarrow \quad y_p = -te^t \quad \Longrightarrow \quad y = c_1 e^t + c_2 e^{2t} + c_3 e^{-2t} - te^t.$$

4a. First of all, we compute the eigenvalues of the associated matrix, namely

$$A = \begin{bmatrix} a & 1 \\ 1 & a \end{bmatrix} \quad \Longrightarrow \quad \lambda^2 - 2a\lambda + (a^2-1) = 0 \quad \Longrightarrow \quad \lambda = a \pm 1.$$

If $a < -1$, then both eigenvalues are negative and the zero solution is asymptotically stable. If $a > -1$, then one eigenvalue is positive and the zero solution is unstable. In the remaining case $a = -1$, the eigenvalues are $\lambda = -2, 0$ and the zero solution is stable but not asymptotically stable.

4b. Note that V is positive definite with

$$V^*(x,y) = 2xx' + 2yy' = -2x^2 + 4xy - 2ay^2 = -2(x-y)^2 + 2(1-a)y^2.$$

It easily follows that V is a strict Lyapunov function if and only if $a > 1$.

5a. Since $y = 1$ is a solution, we have $q(t) = 0$. Since $y = t^2$ is a solution, we have

$$2 + 2tp(t) + t^2 q(t) = 0 \quad \Longrightarrow \quad p(t) = -1/t.$$

5b. Note that $\lambda = 1$ is a simple eigenvalue with corresponding eigenvector

$$\boldsymbol{v}_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

There is also a double eigenvalue, namely $\lambda = 2$, with only one eigenvector

$$\boldsymbol{v}_1 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}.$$

This means that we are missing an eigenvector and that the Jordan form is

$$P^{-1}AP = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \Longrightarrow \quad e^{tP^{-1}AP} = \begin{bmatrix} e^{2t} & te^{2t} & \\ & e^{2t} & \\ & & e^t \end{bmatrix}.$$

To actually find the columns of P, we need to solve the equations

$$A\boldsymbol{v}_1 = 2\boldsymbol{v}_1, \qquad A\boldsymbol{v}_2 = \boldsymbol{v}_1 + 2\boldsymbol{v}_2, \qquad A\boldsymbol{v}_3 = \boldsymbol{v}_3.$$

Thus, we can take $\boldsymbol{v}_1, \boldsymbol{v}_3$ to be the eigenvectors above, and we can take $\boldsymbol{v}_2$ so that

$$(A - 2I)\boldsymbol{v}_2 = \boldsymbol{v}_1 \quad \Longrightarrow \quad \boldsymbol{v}_2 = \begin{bmatrix} x \\ 1 + x/2 \\ 1/2 \end{bmatrix} \quad \Longrightarrow \quad \boldsymbol{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 1/2 \end{bmatrix},$$

for instance. Then these three vectors give the columns of P and we get

$$e^{tA} = P \cdot e^{tP^{-1}AP} \cdot P^{-1} = \begin{bmatrix} e^t & 2e^{2t} - 2e^t & 4te^{2t} - 4e^{2t} + 4e^t \\ 0 & e^{2t} & 2te^{2t} \\ 0 & 0 & e^{2t} \end{bmatrix}.$$