UNIVERSITY OF DUBLIN

XMA23261

TRINITY COLLEGE

Faculty of Engineering, Mathematics and Science

school of mathematics

SF Maths, JS TSM **Trinity Term 2010**

MA2326 Ordinary Differential Equations

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Attempt FOUR questions. All questions are weighted equally.
Log tables are available from the invigilators, if required.

1. [5 points each] Prove each of the following statements.

   (a) Every solution of $x''(t) + x(t) = 0$ is bounded.

   (b) Every solution of $x''(t) + x(t) = \sin t$ is unbounded.

   (c) Every solution of $x''(t) + x(t) = \sin(2t)$ is bounded.

   (d) The initial value problem $tx'(t) = x(t)$, $x(0) = 1$ has no solutions.

   (e) The initial value problem $tx'(t) = x(t)$, $x(0) = 0$ has infinitely many solutions.

2. [25 points]

   (a) [10 points] Determine the unique solution $y = y(t)$ of the initial value problem

$$y' - \frac{ty}{t^2+1} = t, \qquad y(0) = 0.$$

   (b) [5 points] Show that $y = y(t)$ is a nonzero solution of the nonlinear ODE

$$y' + P(t)y = Q(t)y^2$$

   if and only if $z = y^{-1}$ is a solution of the first-order linear ODE

$$z' = P(t)z - Q(t).$$

   (c) [10 points] Find all nonzero solutions $y = y(t)$ of the nonlinear ODE

$$ty' + y = y^2 \log t, \qquad t > 0.$$

3. [25 points]

   (a) [15 points] Check that $y_1(t) = e^t$ is a solution of the second-order ODE

$$(t+1)y'' - (t+2)y' + y = 0$$

   and then use this fact to find all solutions of the ODE.

   (b) [10 points] Find all solutions $y = y(t)$ of the third-order ODE

$$y''' - y'' - 4y' + 4y = 3e^t.$$

4. [25 points]

   (a) [15 points] Let $a \in \mathbb{R}$ be fixed and consider the autonomous linear system

$$x'(t) = ax(t) + y(t), \qquad y'(t) = x(t) + ay(t).$$

   For which values of a is the zero solution stable? asymptotically stable?

   (b) [10 points] Let $a \in \mathbb{R}$ be fixed and consider the autonomous linear system

$$x'(t) = -x(t) + y(t), \qquad y'(t) = x(t) - ay(t).$$

   For which values of a is $V(x, y) = x^2 + y^2$ a strict Lyapunov function?

5. [25 points]

   (a) [10 points] Find a second-order linear ODE of the form

$$y''(t) + p(t)y'(t) + q(t)y(t) = 0, \qquad t > 0$$

   such that its solutions are given by $y(t) = C_1 + C_2 t^2$ for some $c_1, c_2 \in \mathbb{R}$.

   (b) [15 points] Compute the matrix exponential e^{tA} in the case that

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 2 \end{bmatrix}.$$