2009 final exam

1a. $x(t) = c_1 e^{-t} + c_2 e^{-2t}$ is both bounded and going to zero.

1b. $x(t) = c_1 e^{-t} \sin(t\sqrt{2}\,) + c_2 e^{-t} \cos(t\sqrt{2}\,)$ is both bounded and going to zero.

1c. $x(t) = c_1 e^{3t} \sin t + c_2 e^{3t} \cos t$ could be unbounded and not going to zero.

1d. $x(t) = c_1 e^{-3t} \sin t + c_2 e^{-3t} \cos t$ is both bounded and going to zero.

1e. $x(t) = c_1 e^{-t} + c_2 \sin t + c_3 \cos t$ is bounded but need not go to zero.

2a. The given ODE is first-order linear with integrating factor

$$\mu = \exp\left(-\int \frac{t\,dt}{t^2+1}\right) = \exp\left(-\frac{1}{2}\log(t^2+1)\right) = (t^2+1)^{-1/2}.$$

Multiplying by this factor and integrating, we now get

$$(\mu y)' = \frac{t}{\sqrt{t^2+1}} \quad \Longrightarrow \quad \mu y = \int \frac{2t\,dt}{2\sqrt{t^2+1}} = \int \frac{du}{2\sqrt{u}} = \sqrt{u} + C$$

using the substitution $u = t^2 + 1$. This also implies that

$$\frac{y}{\sqrt{t^2+1}} = \sqrt{t^2+1} + C \quad \Longrightarrow \quad y = t^2 + 1 + C\sqrt{t^2+1}.$$

Since we need to have $y(0) = 0$, it easily follows that $y = t^2 + 1 - \sqrt{t^2+1}$.

2b. The given ODE is separable, so we may separate variables to get

$$\frac{dy}{dt} = (1+2t)(1+y) \quad \Longrightarrow \quad \int \frac{dy}{1+y} = \int (1+2t)\,dt$$
$$\Longrightarrow \quad \log|1+y| = t + t^2 + C \quad \Longrightarrow \quad y = Ce^{t+t^2} - 1.$$

Since we need to have $y(0) = 2$, it easily follows that $y = 3e^{t+t^2} - 1$.

2c. Since $y = tz$, we have $y' = z + tz'$, hence $y' = f(y/t)$ if and only if $tz' = f(z) - z$.

3a. The fact that $y_1 = e^t$ is a solution follows by the computation

$$(t-1)y_1'' - ty_1' + y_1 = (t - 1 - t + 1)e^t = 0.$$

We now use reduction of order to find a second solution of the form $y_2 = e^t v$. Since

$$y_2 = e^t v, \qquad y_2' = e^t(v + v'), \qquad y_2'' = e^t(v + 2v' + v''),$$

we see that $y_2 = e^t v$ is also a solution, provided that

$$0 = (t-1)y_2'' - ty_2' + y_2 = (t-1)e^t v'' + (t-2)e^t v'.$$

Dividing through by $(t-1)e^t$ gives a linear ODE with integrating factor

$$\mu = \exp\left(\int \frac{t-2}{t-1}\,dt\right) = \exp\left(\int 1 - \frac{1}{t-1}\,dt\right) = \frac{e^t}{t-1}.$$

We now multiply by this factor and we integrate to get

$$(\mu v')' = 0 \quad\Longrightarrow\quad v' = C_1/\mu = C_1 e^{-t}(t-1).$$

Using this fact and an integration by parts, we conclude that

$$v = C_1 \int e^{-t}(t-1)\,dt = -C_1 te^{-t} + C_2 \quad\Longrightarrow\quad y_2 = -C_1 t + C_2 e^t.$$

3b. To find the homogeneous solution y_h, we note that

$$\begin{aligned} \lambda^3 - \lambda^2 - \lambda + 1 = 0 \quad &\Longrightarrow \quad \lambda^2(\lambda-1) - (\lambda-1) = 0 \\ &\Longrightarrow \quad (\lambda-1)(\lambda-1)(\lambda+1) = 0 \\ &\Longrightarrow \quad y_h = c_1 e^t + c_2 te^t + c_3 e^{-t}. \end{aligned}$$

Based on this fact, we now look for a particular solution of the form

$$y_p = Ate^{-t}.$$

Differentiating this expression three times, one finds that

$$y_p' = Ae^{-t}(1-t), \qquad y_p'' = Ae^{-t}(t-2), \qquad y_p''' = Ae^{-t}(3-t).$$

In particular, $y_p''' - y_p'' - y_p' + y_p = 4Ae^{-t}$ and this implies

$$A = 1 \quad\Longrightarrow\quad y_p = te^{-t} \quad\Longrightarrow\quad y = c_1 e^t + c_2 te^t + c_3 e^{-t} + te^{-t}.$$

4a. First of all, we compute the eigenvalues of the associated matrix, namely

$$A = \begin{bmatrix} a & 1 \\ 1 & a \end{bmatrix} \quad\Longrightarrow\quad \lambda^2 - 2a\lambda + (a^2-1) = 0 \quad\Longrightarrow\quad \lambda = a \pm 1.$$

If $a < -1$, then both eigenvalues are negative and the zero solution is asymptotically stable. If $a > -1$, then one eigenvalue is positive and the zero solution is unstable. In the remaining case $a = -1$, the eigenvalues are $\lambda = -2, 0$ and the zero solution is stable but not asymptotically stable.

4b. Note that V is positive definite with

$$V^*(x,y) = 2xx' + 2yy' = -2x^2 + 4xy - 2ay^2 = -2(x-y)^2 + 2(1-a)y^2.$$

It easily follows that V is a strict Lyapunov function if and only if $a > 1$.

5a. Consider any two solutions of the ODE, say y_i and y_j. Subtracting the equations

$$y_i'' + py_i' + qy_i = r,$$
$$y_j'' + py_j' + qy_j = r,$$

one finds that $y_i - y_j$ satisfies the associated homogeneous ODE. Thus, each of

$$y_2 - y_1 = 2e^{-t}, \qquad y_3 - y_1 = e^{2t}$$

is a solution of the homogeneous ODE, so these two functions must actually generate the space of all solutions. In other words, every solution of the homogeneous ODE

$$y'' + py' + qy = 0$$

has the form $y = c_1e^{-t} + c_2e^{2t}$. This means that $\lambda_1 = -1$ and $\lambda_2 = 2$ are such that

$$\lambda^2 + p\lambda + q = 0 \quad \Longrightarrow \quad p = -(\lambda_1 + \lambda_2) = -1, \qquad q = \lambda_1\lambda_2 = -2.$$

Since $y_1 = e^t$ is a solution of the non-homogeneous ODE, we also have

$$r = y_1'' + py_1' + qy_1 = e^t - e^t - 2e^t = -2e^t.$$

5b. We examine the roots of the associated quadratic equation

$$\lambda^2 + b\lambda + c = 0 \quad \Longrightarrow \quad \lambda = \frac{-b \pm \sqrt{b^2 - 4c}}{2}.$$

- If these roots are real, then they are both negative because

  $$b^2 - 4c < b^2 \quad \Longrightarrow \quad \sqrt{b^2 - 4c} < b.$$

  Since the solution is either $c_1e^{\lambda_1 t} + c_2e^{\lambda_2 t}$ or $c_1e^{\lambda_1 t} + c_2te^{\lambda_1 t}$ and the λ_i's are negative, the solution does go to zero as $t \to \infty$.

- If the roots are not real, then we have $\lambda = -\frac{b}{2} \pm i\gamma$ for some $\gamma \in \mathbb{R}$ and so

  $$y(t) = c_1e^{-bt/2}\sin(\gamma t) + c_2e^{-bt/2}\cos(\gamma t).$$

  Since $b > 0$ by assumption, however, the solution still goes to zero as $t \to \infty$.