2008 final exam

1a. Only (ii), (iii) and (v) are linear. Only (ii) and (v) are homogeneous.

1b. When it comes to the first matrix, its eigenvalues are given by

$$\lambda^2 - (\operatorname{tr} A)\lambda + \det A = 0 \quad \Longrightarrow \quad \lambda^2 - 2\lambda + 1 = 0 \quad \Longrightarrow \quad \lambda = 1.$$

Since there is only one eigenvalue, we need to look at the nullspace of $A - \lambda I$, namely

$$A - I = \begin{bmatrix} 2 & -1 \\ 4 & -2 \end{bmatrix} \quad \Longrightarrow \quad \mathscr{N}(A - I) = \left\{ \begin{bmatrix} x \\ 2x \end{bmatrix} : x \in \mathbb{R} \right\}.$$

Pick any vector $\boldsymbol{v}_2$ which is not in this nullspace and let $\boldsymbol{v}_1 = (A - I)\boldsymbol{v}_2$, say

$$\boldsymbol{v}_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad \boldsymbol{v}_1 = (A - I)\boldsymbol{v}_2 = \begin{bmatrix} 2 & -1 \\ 4 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}.$$

Then these two vectors give rise to a Jordan basis and we have

$$P = \begin{bmatrix} 2 & 1 \\ 4 & 0 \end{bmatrix} \quad \Longrightarrow \quad P^{-1}AP = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \quad \Longrightarrow \quad e^{tP^{-1}AP} = \begin{bmatrix} e^t & te^t \\ 0 & e^t \end{bmatrix}$$
$$\Longrightarrow \quad e^{tA} = P \cdot e^{tP^{-1}AP} \cdot P^{-1} = e^t \begin{bmatrix} 1 + 2t & -t \\ 4t & 1 - 2t \end{bmatrix}.$$

- When it comes to the second matrix, note that $B = A + I$ is a matrix whose entries are all equal to 1. It easily follows that $B^2 = 3B$, hence

$$B^3 = 3B^2 = 3^2 B, \qquad B^4 = 3^2 B^2 = 3^3 B,$$

and so on. Using induction and the definition of the exponential, we now get

$$e^{tB} = \sum_{n=0}^{\infty} \frac{t^n B^n}{n!} = I + \sum_{n=1}^{\infty} \frac{t^n 3^{n-1} B}{n!} = I + \frac{1}{3} \sum_{n=1}^{\infty} \frac{(3t)^n}{n!} B = I + \frac{e^{3t} - 1}{3} B.$$

Since $B = A + I$ by above, this also implies

$$e^{tA} = e^{tB} e^{-tI} = \left(I + \frac{e^{3t} - 1}{3} B \right) e^{-t} = \frac{e^{2t} + 2e^{-t}}{3} I + \frac{e^{2t} - e^{-t}}{3} A.$$

That is, all diagonal entries of e^{tA} are $\frac{e^{2t}+2e^{-t}}{3}$ and all other entries are $\frac{e^{2t}-e^{-t}}{3}$.

2a. There are obviously many different choices that will do. For instance, take $t = 1$ and

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

Then A is a Jordan block and B is diagonal, so one can easily compute

$$e^A e^B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & e \end{bmatrix} = \begin{bmatrix} 1 & e \\ 0 & e \end{bmatrix},$$
$$e^B e^A = \begin{bmatrix} 1 & 0 \\ 0 & e \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & e \end{bmatrix}.$$

To compute the exponential of $A + B$, we now use the fact that

$$A + B = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \quad \Longrightarrow \quad (A+B)^n = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} = A + B$$

for each $n \geq 1$. In view of the definition of the exponential, this gives

$$e^{A+B} = \sum_{n=0}^{\infty} \frac{1}{n!} (A+B)^n = I + \sum_{n=1}^{\infty} \frac{1}{n!} (A+B)$$
$$= I + (e-1)(A+B) = \begin{bmatrix} 1 & e-1 \\ 0 & e \end{bmatrix}$$

so the three matrices $e^A e^B$, $e^B e^A$ and e^{A+B} are all distinct.

2b. Differentiating the first equation twice, one finds that

$$(A+B)^2 e^{t(A+B)} = A^2 e^{tA} e^{tB} + 2A e^{tA} B e^{tB} + e^{tA} B^2 e^{tB}.$$

Setting $t = 0$ and simplifying, one thus arrives at

$$A^2 + AB + BA + B^2 = A^2 + 2AB + B^2 \quad \Longrightarrow \quad BA = AB.$$

In fact, the same argument applies, if one starts with any of the given equations.

2c. To find the homogeneous solution x_h, we note that

$$\lambda^2 + 1 = 0 \quad \Longrightarrow \quad \lambda = \pm i \quad \Longrightarrow \quad x_h = c_1 \sin t + c_2 \cos t.$$

Based on this fact, we now look for a particular solution of the form

$$x_p = At^2 + Bt + C \quad \Longrightarrow \quad x_p' = 2At + B \quad \Longrightarrow \quad x_p'' = 2A.$$

To say that x_p is a solution of the given ODE is to say that

$$t^2 = x_p'' + x_p = At^2 + Bt + C + 2A.$$

This gives $A = 1$, $B = 0$ and also $C = -2A = -2$, hence

$$x_p = t^2 - 2 \quad \Longrightarrow \quad x = c_1 \sin t + c_2 \cos t + t^2 - 2.$$

Once we now note that $x(0) = c_2 - 2$ and $x'(0) = c_1$, it easily follows that

$$c_2 - 2 = \xi, \quad c_1 = \eta \quad \Longrightarrow \quad x = \eta \sin t + (\xi + 2) \cos t + t^2 - 2.$$

3a. Both equations are first-order linear with integrating factor

$$\mu = \exp\left(\int 2t\,dt\right) = e^{t^2}.$$

When it comes to the first equation, one has

$$x' + 2tx = 0 \quad\Longrightarrow\quad (\mu x)' = 0 \quad\Longrightarrow\quad x = C/\mu = Ce^{-t^2}.$$

When it comes to the second equation, one similarly has

$$\begin{aligned} x' + 2tx = t \quad &\Longrightarrow\quad (\mu x)' = te^{t^2} \\ &\Longrightarrow\quad \mu x = \frac{1}{2}e^{t^2} + C \quad\Longrightarrow\quad x = \frac{1}{2} + Ce^{-t^2}. \end{aligned}$$

3b. Using reduction of order, one can find a second solution $x_2 = x_1 v$ by solving

$$v'' + \left(\frac{2x_1'}{x_1} - \frac{4t^3+2t}{t^4+t^2+4}\right)v' = 0.$$

Noting that

$$\int \frac{2x_1'}{x_1} - \frac{4t^3+2t}{t^4+t^2+4}\;dt = 2\log x_1 - \log(t^4+t^2+4),$$

we see that an integrating factor is given by

$$\mu = \exp\left(\int \frac{2x_1'}{x_1} - \frac{4t^3+2t}{t^4+t^2+4}\;dt\right) = \frac{x_1^2}{t^4+t^2+4} = \frac{t^2(t^2-4)^2}{t^4+t^2+4}.$$

We now multiply by this factor and we integrate to get

$$(\mu v')' = 0 \quad\Longrightarrow\quad v' = \frac{C_1}{\mu} = \frac{C_1(t^4+t^2+4)}{t^2(t^2-4)^2}.$$

To integrate the function on the right, one needs to use partial fractions to write

$$\frac{t^4+t^2+4}{t^2(t^2-4)^2} = \frac{1/4}{t^2} + \frac{3/8}{(t-2)^2} + \frac{3/8}{(t+2)^2}.$$

Integrating the last equation and simplifying, we conclude that

$$v = -C_1\left(\frac{1/4}{t} + \frac{3/8}{t-2} + \frac{3/8}{t+2}\right) + C_2 = -\frac{C_1(t^2-1)}{t(t^2-4)} + C_2.$$

In particular, the desired solution $x_2 = x_1 v$ of the given ODE is

$$x_2 = x_1 v = -C_1(t^2-1) + C_2 t(t^2-4).$$

4a. I will not ask for any definitions.

4b. Indeed, $E(t) = x'(t)^2 + x(t)^4$ is such that $E'(t) = 2x'(x'' + 2x^3) = 0$.

4c. Since $x'(t)^2 + x(t)^4 = E(t) = E(0)$, both $x'(t)$ and $x(t)$ are bounded at all times.

4d. Write the second-order equation $x'' + 2x^3 = 0$ as a first-order system, namely

$$x' = y, \qquad y' = x'' = -2x^3.$$

According to part (b), this system has $V(x, y) = x^4 + y^2$ as a Lyapunov function and so the origin is stable by the Lyapunov theorem.

4e. It is not asymptotically stable because $V(x, y) = x^4 + y^2$ is both positive definite and conserved. That is, every solution which converges to the origin must satisfy

$$V(x_0, y_0) = \lim_{t\to\infty} V(x(t), y(t)) = V(0, 0) = 0 \quad \Longrightarrow \quad (x_0, y_0) = (0, 0)$$

and so solutions which start out near the origin cannot really converge to it.

5a. I will not ask for any definitions.

5b. This is a linear system whose eigenvalues are given by

$$\lambda^2 - (\operatorname{tr} A)\lambda + \det A = 0 \quad \Longrightarrow \quad \lambda^2 - 3a\lambda + 2a^2 = 0 \quad \Longrightarrow \quad \lambda = a, 2a$$

so the origin is unstable when $a > 0$ and asymptotically stable when $a < 0$. To deal with the remaining case $a = 0$, we note that

$$A = \begin{bmatrix} 2b & -b \\ 4b & -2b \end{bmatrix} \quad \Longrightarrow \quad A^2 = 0 \quad \Longrightarrow \quad e^{tA} = I + tA.$$

When $a = 0 \neq b$, the entries of $e^{tA}\boldsymbol{x}_0$ are thus unbounded and the origin is unstable. When $a = b = 0$, we have $e^{tA}\boldsymbol{x}_0 = \boldsymbol{x}_0$ so the origin is stable but not asymptotically.

5c. Note that $V(x, y) = x^2 + y^2$ measures distance from the origin and that

$$V^*(x, y) = 2xx' + 2yy' = 2ax^2(x^2 + y^2) + 2ay^2(x^2 + y^2) = 2a(x^2 + y^2)^2.$$

If $a < 0$, this makes V a strict Lyapunov function and so the origin is asymptotically stable. If $a = 0$, then distance from the origin is conserved, so the origin is stable but not asymptotically. If $a > 0$, finally, then the computation above gives

$$\frac{dV}{dt} = 2aV^2 \quad \Longrightarrow \quad \int V^{-2}\, dV = 2a \int dt \quad \Longrightarrow \quad -V^{-1} = 2at - V_0^{-1}$$
$$\Longrightarrow \quad V(t) = (1/V_0 - 2at)^{-1}.$$

In particular, $V(t)$ becomes arbitrarily large as $t \to \frac{1}{2aV_0}$ so the origin is unstable.