UNIVERSITY OF DUBLIN

XMA2161

TRINITY COLLEGE

FACULTY OF ENGINEERING, MATHEMATICS
AND SCIENCE

SCHOOL OF MATHEMATICS

SF Mathematics
SF Theoretical Physics

Trinity Term 2008

COURSE 216

Tuesday, 20 May 2008 Sports Hall 9:30-11:30

John Stalker

Attempt four questions.
Log tables are available from the invigilators, if required.

Non-programmable calculators are permitted for this examination,—please indicate the make and model of your calculator on each answer book used.

1. (25 points)

   (a) (1 point each) Which of the following differential equations or systems are linear? Of those that are linear, which are homogeneous?

      i. Lotka-Volterra

$$\begin{aligned} x'(t) &= x(t)(\alpha - \beta y(t)) \\ y'(t) &= -y(t)(\gamma - \delta y(t)) \end{aligned}$$

      ii. Legendre

$$(1 - t^2)x''(t) - 2tx(t) + l(l+1)x(t) = 0$$

      iii. Sinusoidally forced harmonic oscillator

$$mx''(t) + 2\gamma x'(t) + kx(t) = A\cos(\omega t - \varphi)$$

      iv. Rigid body

$$\begin{aligned} I_1\Omega_1'(t) &= (I_2 - I_3)\Omega_2(t)\Omega_3(t) \\ I_2\Omega_2'(t) &= (I_3 - I_1)\Omega_3(t)\Omega_1(t) \\ I_3\Omega_3'(t) &= (I_1 - I_2)\Omega_1(t)\Omega_2(t) \end{aligned}$$

      v. Bessel

$$t^2x''(t) + tx'(t) + (t^2 - \nu^2)x(t) = 0$$

   (b) (10 points each) Compute $\exp(tA)$ where

      i.

$$A = \begin{pmatrix} 3 & -1 \\ 4 & -1 \end{pmatrix}$$

      ii.

$$A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

2. (25 points)

(a) (8 points) In general, i.e. for randomly chosen real numbers t and square matrices A and B, none of the following is true

$$\exp(tA+tB) = \exp(tA)\exp(tB)$$
$$\exp(tA+tB) = \exp(tB)\exp(tA)$$
$$\exp(tA)\exp(tB) = \exp(tB)\exp(tA)$$

Give an example of a specific choice of t, A and B such that none of these equations is satisfied. You should compute $\exp(tA+tB)$, $\exp(tA)\exp(tB)$ and $\exp(tB)\exp(tA)$ for your choices of t, A and B.

(b) (8 points) Show that if, for given A and B, one of the equations above holds for all t then $AB = BA$.

(c) (9 points) Find the solution to the initial value problem

$$x''(t) + x(t) = t^2 \qquad x(0) = \xi \qquad x'(0) = \eta$$

3. (25 points)

(a) (6 points each) Find the general solution to

i.

$$x'(t) + 2tx(t) = 0$$

ii.

$$x'(t) + 2tx(t) = t$$

(b) (13 points) A particular solution to the differential equation

$$(t^4+t^2+4)x''(t) - (4t^3+2t)x'(t) + (6t^2+8)x(t) = 0$$

is

$$x_1(t) = t^3 - 4t.$$

Find the general solution.

4. (25 points)

   (a) (1 point each) Define the following:

      i. Autonomous system

      ii. Equilibrium of an autonomous system

      iii. Stable equilibrium of an autonomous system

      iv. Asymptotically stable equilibrium of an autonomous system

      v. Strictly stable equilibrium of an autonomous system

   (b) (4 points) Prove that

$$x'(t)^2 + x(t)^4$$

   is an invariant of

$$x''(t) + 2x(t)^3 = 0$$

   (c) (8 points) Prove that all solutions of $x''(t) + 2x^3(t) = 0$ are bounded.

   (d) (4 points) Is 0 a stable equilibrium of $x''(t) + 2x^3(t) = 0$? Why or why not?

   (e) (4 points) Is 0 an asymptotically stable equilibrium of $x''(t) + 2x^3(t) = 0$? Why or why not?

5. (25 points)

(a) (2 points each) Define the following:

i. Lyapunov function

ii. Strict Lyapunov function

iii. Linearisation

(b) (8 points) For which values of a and b is $(0,0)$ a stable equilibrium of the autonomous system

$$x'(t) = (5a + 2b)x(t) - (2a + b)y(t)$$
$$y'(t) = (6a + 4b)x(t) - (2a + 2b)y(t)$$

For what values of a and b is it a strictly stable equilibrium?

(c) (11 points) For which values of a is $(0,0)$ a stable equilibrium of the autonomous system

$$x'(t) = a(x(t)^2 + y(t)^2)x(t) - y(t)$$
$$y'(t) = x(t) + a(x(t)^2 + y(t)^2)y(t)$$

For what values of a is it a strictly stable equilibrium?