2007 final exam

1a. $x(t) = c_1 e^{-t} \sin(t\sqrt{2}\,) + c_2 e^{-t} \cos(t\sqrt{2}\,)$ is both bounded and going to zero.

1b. $x(t) = c_1 e^{t} \sin(t\sqrt{2}\,) + c_2 e^{t} \cos(t\sqrt{2}\,)$ could be unbounded and not going to zero.

1c. $x(t) = c_1 \sin(2t) + c_2 \cos(2t)$ is bounded but need not go to zero.

1d. $x(t) = c_1 e^{2t} + c_2 e^{-2t}$ could be unbounded and not going to zero.

1e. $x(t) = c_1 + c_2 t + c_3 e^{-t}$ could be unbounded and not going to zero.

2a. It is defined by the series $e^A = \sum_{n=0}^{\infty} \frac{A^n}{n!}$ for each square matrix A.

2b. It is not generally true that $e^{A+B} = e^A e^B$.

2c. The given identity is of the form $A = PJP^{-1}$, where J is diagonal, and thus

$$e^{tA} = e^{tPJP^{-1}} = Pe^{tJ}P^{-1} = \begin{bmatrix} 15e^{-t} - 14e^{2t} & 21e^{-t} - 21e^{2t} \\ 10e^{2t} - 10e^{-t} & 15e^{2t} - 14e^{-t} \end{bmatrix}.$$

As for the solution to the initial value problem, this is given by $\boldsymbol{x}(t) = e^{tA}\boldsymbol{x}(0)$.

3a. Indeed, $A' = 2xx' + 2yy' = 2xyz - 2xyz = 0$ and similarly $B' = 2xx' + 2zz' = 0$.

3b. Indeed, we have $x(t)^2 + y(t)^2 = x(0)^2 + y(0)^2$ and also $x(t)^2 + z(t)^2 = x(0)^2 + z(0)^2$.

3c. Let $V(x, y, z) = A + B = 2x^2 + y^2 + z^2$. Then V is positive definite with $V^* = 0$, so the stability of the zero solution follows by the Lyapunov theorem.

4. Using reduction of order, one can start with a solution x_1 of the ODE

$$x'' + p(t)x' + q(t)x = 0$$

and construct a second solution $x_2 = x_1 v$ by solving the ODE

$$v'' + \left(\frac{2x_1'}{x_1} + p\right) v' = 0.$$

In this case, we have

$$\int \frac{2x_1'}{x_1} + p(t)\, dt = \int \frac{2x_1'}{x_1} + \frac{2 - 2t}{t^2 - 2t - 1}\, dt = 2\log x_1 - \log(t^2 - 2t - 1)$$

so an integrating factor is given by

$$\mu = \exp\left(\int \frac{2x_1'}{x_1} + p(t)\, dt\right) = \frac{x_1^2}{t^2 - 2t - 1} = \frac{(t^2 + 1)^2}{t^2 - 2t - 1}.$$

Multiplying by this factor and integrating, we now get

$$(\mu v')' = 0 \quad \Longrightarrow \quad v' = \frac{C_1}{\mu} \quad \Longrightarrow \quad v = C_1 \int \frac{t^2 - 2t - 1}{(t^2+1)^2}\, dt.$$

In order to compute the last integral, we first integrate by parts to get

$$\int (t^2+1)^{-1}\, dt = t(t^2+1)^{-1} + \int 2t^2(t^2+1)^{-2}\, dt.$$

Using this formula and a little bit of algebra, we find that

$$\int \frac{t^2-1}{(t^2+1)^2}\, dt = \int \frac{2t^2}{(t^2+1)^2} - \frac{1}{t^2+1}\, dt = -\frac{t}{t^2+1} + C,$$

while the substitution $u = t^2 + 1$ gives

$$\int \frac{2t}{(t^2+1)^2}\, dt = \int u^{-2}\, du = -u^{-1} + C = -\frac{1}{t^2+1} + C.$$

Once we now combine the last two equations, we arrive at

$$v = C_1 \int \frac{t^2 - 1 - 2t}{(t^2+1)^2}\, dt = -\frac{C_1 t}{t^2+1} + \frac{C_1}{t^2+1} + C_2.$$

Thus, the desired solution $x_2 = x_1 v$ of the given ODE is

$$x_2 = (t^2+1)v = -C_1 t + C_1 + C_2(t^2+1) = C_1(1-t) + C_2(t^2+1).$$

5a. I will not ask for any definitions.

5b. Indeed, V is positive definite with $V^*(x,y) = 2xx' + 2yy' = -2x^2 - 2y^2$.

6. Power series solutions are no longer covered in 216.