2 First-order linear ODE

Definition 2.1 (Linear & homogeneous). An ODE is called linear, if it has the form

$$a_n(x)y^{(n)} + \ldots + a_1(x)y' + a_0(x)y = b(x).$$

A linear ODE is called homogeneous, if it has $y = 0$ as a solution, namely if $b(x) = 0$.

Theorem 2.2 (First-order linear ODE). To solve the first-order linear ODE

$$y'(x) + P(x)y(x) = Q(x),$$

one introduces the integrating factor $\mu(x) = \exp\left(\int P(x)\,dx\right)$ and then notes that

$$y'(x) + P(x)y(x) = Q(x) \quad\iff\quad \mu(x)y(x) = \int \mu(x)Q(x)\,dx.$$

Example 2.3. Consider the first-order linear equation

$$y'(x) + 3y(x) = 4e^x.$$

According to the theorem above, an integrating factor for this equation is

$$\mu(x) = \exp\left(\int 3\,dx\right) = e^{3x}.$$

Multiplying the ODE by this factor, we now get a perfect derivative on the left, namely

$$y' + 3y = 4e^x \quad\iff\quad y'e^{3x} + 3ye^{3x} = 4e^{4x} \quad\iff\quad (ye^{3x})' = 4e^{4x}.$$

Next, we integrate the rightmost equation and we solve for y to conclude that

$$ye^{3x} = \int 4e^{4x}\,dx = e^{4x} + C \quad\iff\quad y = e^x + Ce^{-3x}.$$

Example 2.4. Consider the first-order linear equation

$$y'(t) + \frac{y(t)}{t} = 2.$$

In this case, an integrating factor is given by

$$\mu(t) = \exp\left(\int \frac{1}{t}\,dt\right) = \exp(\log t) = t.$$

Note that the integral in brackets was not computed in its most general form $\log|t| + C$ because we only needed one integrating factor, so we chose the simplest one possible. Once again, multiplication by $\mu(t)$ should give a perfect derivative on the left, namely

$$y' + \frac{y}{t} = 2 \quad\iff\quad ty' + y = 2t \quad\iff\quad (yt)' = 2t.$$

Integrating with respect to t, we now get $yt = t^2 + C$ and this implies that $y = t + C/t$.