14 Stability of nonlinear systems

Theorem 14.1 (Stability of nonlinear systems). Consider the system

$$x'(t) = f(x, y), \qquad y'(t) = g(x, y),$$

where f, g are differentiable with continuous partial derivatives and they both vanish at the point (x_0, y_0). Let J denote the Jacobian matrix at that point, namely

$$J = \begin{bmatrix} f_x(x_0, y_0) & f_y(x_0, y_0) \\ g_x(x_0, y_0) & g_y(x_0, y_0) \end{bmatrix}.$$

If all eigenvalues of J have negative real part, then (x_0, y_0) is asymptotically stable. And if some eigenvalue of J has positive real part, then (x_0, y_0) is unstable.

Remark 14.2. This theorem provides no conclusions in the case that some eigenvalue of J has zero real part. Should that case arise, stability can be determined either by finding a Lyapunov function for the system or else by solving the system explicitly.

Example 14.3. We determine the stability properties for the critical points of the system

$$x'(t) = 1 - y, \qquad y'(t) = x^2 - y^2.$$

In this case, the critical points satisfy $y = 1$ and $x^2 = y^2$, so the only such points are

$$A(-1, 1), \qquad B(1, 1).$$

To determine their stability properties, we now look at the Jacobian matrix

$$J = \begin{bmatrix} f_x & f_y \\ g_x & g_y \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 2x & -2y \end{bmatrix}.$$

At the critical point $A(-1, 1)$, the eigenvalues of this matrix are

$$J = \begin{bmatrix} 0 & -1 \\ -2 & -2 \end{bmatrix} \quad \Longrightarrow \quad \lambda^2 + 2\lambda - 2 = 0 \quad \Longrightarrow \quad \lambda = -1 \pm \sqrt{3}$$

and one of those is positive, so A is unstable. At the critical point $B(1, 1)$, we have

$$J = \begin{bmatrix} 0 & -1 \\ 2 & -2 \end{bmatrix} \quad \Longrightarrow \quad \lambda^2 + 2\lambda + 2 = 0 \quad \Longrightarrow \quad \lambda = -1 \pm i,$$

hence both eigenvalues have negative real part and B is asymptotically stable.