13 Lyapunov stability

Definition 13.1 (Positive definite). We say that $V(x, y)$ is positive definite, if $V(x, y) \geq 0$ at all points and if equality holds only at the origin.

Definition 13.2 (Lyapunov functions). Suppose that $V(x, y)$ is continuous and positive definite. Then we say that V is a Lyapunov function for the system

$$x'(t) = f(x, y), \qquad y'(t) = g(x, y), \tag{13.1}$$

if $V^* = V_x f + V_y g$ is such that $V^*(x, y) \leq 0$ in some open region around the origin. We say that V is a strict Lyapunov function, if we also have $V^*(x, y) = 0$ only at the origin.

Theorem 13.3 (Lyapunov stability theorem). Consider the system (13.1), where f, g are differentiable with continuous partial derivatives and they both vanish at the origin. Then the zero solution is stable, if the system has a Lyapunov function, and it is asymptotically stable, if the system has a strict Lyapunov function.

Example 13.4. We show that the zero solution is a stable solution of the system

$$x'(t) = xy^2 - x, \qquad y'(t) = -2x^2 y.$$

Note that $V(x, y) = ax^2 + by^2$ is positive definite for any $a, b > 0$. If we can choose a, b such that $V^*(x, y) \leq 0$, then stability will follow by the theorem above. Noting that

$$V^*(x, y) = 2axx' + 2byy' = (2a - 4b)x^2y^2 - 2ax^2,$$

we can thus take $a = 2b$ for any $a > 0$, in which case $V^*(x, y) = -4bx^2 \leq 0$, as needed.

Example 13.5. Consider a slight variation of the previous system, namely

$$x'(t) = xy^2 - x, \qquad y'(t) = -y - 2x^2 y.$$

Taking $V(x, y) = ax^2 + by^2$ and $a = 2b$ as before, we now get

$$V^*(x, y) = 2axx' + 2byy' = (2a - 4b)x^2y^2 - 2ax^2 - 2by^2 = -4bx^2 - 2by^2.$$

In particular, $V^*(x, y) \leq 0$ at all points and equality holds only at the origin, so V is a strict Lyapunov function and the zero solution is asymptotically stable.