

## 12 Stability of linear systems

**Definition 12.1.** An autonomous system of ODEs is one that has the form  $\mathbf{y}' = \mathbf{f}(\mathbf{y})$ . We say that  $\mathbf{y}_0$  is a critical point (or equilibrium point) of the system, if it is a constant solution of the system, namely if  $\mathbf{f}(\mathbf{y}_0) = \mathbf{0}$ .

**Definition 12.2 (Stability).** A critical point is said to be stable, if every solution which is initially close to it remains close to it for all times. It is said to be asymptotically stable, if it is stable and every solution which is initially close to it converges to it as  $t \rightarrow \infty$ .

**Theorem 12.3 (Stability of linear systems).** Consider the  $2 \times 2$  linear system

$$\mathbf{y}' = A\mathbf{y},$$

where  $A$  is an invertible matrix. Then the only critical point is the origin which corresponds to the zero solution. This solution is stable if and only if the eigenvalues of  $A$  are either real and negative or else complex with non-positive real part; and it is asymptotically stable, if the complex eigenvalues of  $A$  have strictly negative real part.

**Example 12.4.** Consider the linear system  $\mathbf{y}' = A\mathbf{y}$  when

$$A = \begin{bmatrix} 3 & 1 \\ 0 & -1 \end{bmatrix}, \quad A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} -1 & 0 \\ 1 & -2 \end{bmatrix}.$$

In the first case,  $\lambda = 3$  is an eigenvalue, so the zero solution is unstable. In the second case, the eigenvalues are  $\lambda = \pm i$ , so the zero solution is stable but not asymptotically stable. In the third case, both eigenvalues are negative, so the zero solution is (asymptotically) stable.

**Example 12.5 (Nonzero critical points).** Consider the linear system

$$x' = 2x - y, \quad y' = 2x - 2$$

whose only critical point is  $(1, 2)$ . To study its stability, we change variables by

$$u = x - 1, \quad v = y - 2 \quad \implies \quad u' = 2u - v, \quad v' = 2u.$$

This gives rise to a linear system of the form  $\mathbf{y}' = A\mathbf{y}$ , where

$$\mathbf{y} = \begin{bmatrix} u \\ v \end{bmatrix}, \quad A = \begin{bmatrix} 2 & -1 \\ 2 & 0 \end{bmatrix}.$$

Since the eigenvalues of  $A$  are  $\lambda = 1 \pm i$ , the zero solution is unstable and thus the critical point  $(1, 2)$  is also unstable. That is, solutions of the system  $\mathbf{y}' = A\mathbf{y}$  which start out close to the origin need not remain close to it for all times, so the same is true for solutions of the original system which start out close to  $(1, 2)$ .