11 Reduction of order

Theorem 11.1 (Reduction of order). Suppose that $y_1(t)$ is a nonzero solution of

$$y'' + P(t)y' + Q(t)y = 0.$$

Then its multiple $y_2 = vy_1$ is also a solution, provided that v satisfies

$$v'' + \left(\frac{2y_1'}{y_1} + P(t)\right) v' = 0.$$

This fact allows us to find a second solution to the ODE once one solution is known.

Example 11.2. It is easy to check that $y_1(t) = e^{\lambda t}$ is a solution to the ODE

$$y'' - 2\lambda y' + \lambda^2 y = 0.$$

To find a second solution, we look for a multiple of the first, say $y_2 = e^{\lambda t}v$. This gives

$$y_2 = e^{\lambda t}v, \qquad y_2' = \lambda e^{\lambda t}v + e^{\lambda t}v', \qquad y_2'' = \lambda^2 e^{\lambda t}v + 2\lambda e^{\lambda t}v' + e^{\lambda t}v''$$

and thus $y_2 = e^{\lambda t}v$ is also a solution, provided that

$$0 = y_2'' - 2\lambda y_2' + \lambda^2 y_2 = e^{\lambda t}v'' \quad \Longrightarrow \quad v'' = 0.$$

Solving the last equation, we now easily get

$$v' = C_1 \quad \Longrightarrow \quad v = C_1 t + C_2 \quad \Longrightarrow \quad y_2 = C_1 t e^{\lambda t} + C_2 e^{\lambda t}.$$

Example 11.3. It is easy to check that $y_1(t) = t$ is a solution to the ODE

$$y'' - t^{-1}y' + t^{-2}y = 0, \qquad t > 0.$$

To find a second solution, we look for a multiple of the first, say $y_2 = tv$. This gives

$$y_2 = tv, \qquad y_2' = v + tv', \qquad y_2'' = 2v' + tv''$$

and thus $y_2 = tv$ is also a solution, provided that

$$0 = y_2'' - t^{-1}y_2' + t^{-2}y_2 = tv'' + v'.$$

To solve the last equation, we note that its right hand side is a perfect derivative, namely

$$\begin{aligned} tv'' + v' = 0 \quad &\Longrightarrow \quad (tv')' = 0 &&\Longrightarrow \quad v' = C_1/t \\ &\Longrightarrow \quad v = C_1 \log t + C_2 &&\Longrightarrow \quad y_2 = C_1 t \log t + C_2 t. \end{aligned}$$

Once again, this gives two linearly independent solutions including the original solution.