1 Separable ODE

Definition 1.1 (Separable). An ODE is said to be separable, if it has the form

$$y'(x) = f(x) \cdot g(y)$$

for some functions f, g. In order to solve such an ODE, one separates variables to get

$$\frac{dy}{dx} = f(x) \cdot g(y) \quad \Longrightarrow \quad \int \frac{dy}{g(y)} = \int f(x)\, dx.$$

Example 1.2. We use separation of variables to find all functions $u = u(x)$ such that

$$u' = -2xe^u.$$

Following the approach outlined above, we separate variables to get

$$\frac{du}{dx} = -2xe^u \quad \Longrightarrow \quad \int e^{-u}\, du = -\int 2x\, dx \quad \Longrightarrow \quad -e^{-u} = -x^2 + C.$$

Once we now solve the rightmost equation for u, we may conclude that

$$e^{-u} = x^2 - C \quad \Longrightarrow \quad -u = \log(x^2 - C) \quad \Longrightarrow \quad u = -\log(x^2 - C).$$

Example 1.3. We use separation of variables to solve the initial value problem

$$y'(t) = y(t)^2, \qquad y(0) = 1.$$

Assume for the moment that $y(t)$ does not vanish at any point. Then we have

$$\frac{dy}{dt} = y^2 \quad \Longrightarrow \quad \int y^{-2}\, dy = \int dt \quad \Longrightarrow \quad -y^{-1} = t + C$$

and the initial condition $y(0) = 1$ implies that $C = -1$, so we get

$$-\frac{1}{y} = t - 1 \quad \Longrightarrow \quad y = -\frac{1}{t-1}\,.$$

This solution does not vanish at any point, so our computation above is justified.