MA121 Tutorial Problems #8
Solutions

1. *Letting $f(x,y,z) = x^2yz$, find the rate at which f is changing at the point $(1,2,1)$ in the direction of the vector $\mathbf{v} = \langle 1,2,2 \rangle$.*

- To find a unit vector $\mathbf{u}$ in the direction of $\mathbf{v}$, we need to divide $\mathbf{v}$ by its length, namely

$$||\mathbf{v}|| = \sqrt{1^2 + 2^2 + 2^2} = 3 \quad \Longrightarrow \quad \mathbf{u} = \frac{1}{3}\mathbf{v} = \langle 1/3, 2/3, 2/3 \rangle.$$

The desired rate of change is given by the directional derivative $D_{\mathbf{u}}f = \nabla f \cdot \mathbf{u}$. Since

$$\nabla f(x,y,z) = \langle 2xyz, x^2z, x^2y \rangle \quad \Longrightarrow \quad \nabla f(1,2,1) = \langle 4,1,2 \rangle,$$

we may thus conclude that the desired rate of change is

$$D_{\mathbf{u}}f = \nabla f \cdot \mathbf{u} = \frac{1}{3} \cdot 4 + \frac{2}{3} \cdot 1 + \frac{2}{3} \cdot 2 = \frac{10}{3}.$$

2. *Letting $f(x,y) = x^2e^y + xy$, find the direction in which f increases most rapidly at the point $(2,0)$. What is the exact rate of change in that direction?*

- The direction of most rapid increase is that of the gradient vector

$$\nabla f(x,y) = \langle f_x, f_y \rangle = \langle 2xe^y + y, x^2e^y + x \rangle \quad \Longrightarrow \quad \nabla f(2,0) = \langle 4,6 \rangle.$$

To compute the exact rate of change in that direction, we need to find a unit vector $\mathbf{u}$ in the same direction. This amounts to dividing the vector $\nabla f(2,0)$ by its length, namely

$$||\nabla f(2,0)|| = \sqrt{4^2 + 6^2} = \sqrt{52} \quad \Longrightarrow \quad \mathbf{u} = \langle 4/\sqrt{52}, 6/\sqrt{52} \rangle.$$

The desired rate of change is then given by the directional derivative

$$D_{\mathbf{u}}f = \nabla f \cdot \mathbf{u} = \frac{4}{\sqrt{52}} \cdot 4 + \frac{6}{\sqrt{52}} \cdot 6 = \frac{52}{\sqrt{52}} = \sqrt{52} = 2\sqrt{13}.$$

3. *Suppose that $u = x^2 - y^2$, where $x = r\cos\theta$ and $y = r\sin\theta$. Compute u_r and u_θ.*

- Using the definitions of u, x, y together with the chain rule, one finds that

$$\begin{aligned} u_r &= u_x x_r + u_y y_r = 2x\cos\theta - 2y\sin\theta, \\ u_\theta &= u_x x_\theta + u_y y_\theta = -2xr\sin\theta - 2yr\cos\theta. \end{aligned}$$

4. *Suppose that $z = z(r,s,t)$, where $r = u - v$, $s = v - w$ and $t = w - u$. Assuming that all partial derivatives exist, show that $z_u + z_v + z_w = 0$.*

- Using the definitions of z, r, s, t together with the chain rule, one finds that

$$\begin{aligned} z_u &= z_r r_u + z_s s_u + z_t t_u = z_r - z_t, \\ z_v &= z_r r_v + z_s s_v + z_t t_v = -z_r + z_s, \\ z_w &= z_r r_w + z_s s_w + z_t t_w = -z_s + z_t. \end{aligned}$$

Once we now add these three equations, we get the desired identity $z_u + z_v + z_w = 0$.

5. *Suppose that $w = w(u,v)$, where $u = x^{-1} - y^{-1}$ and $v = y^{-1} - z^{-1}$. Assuming that all partial derivatives exist, show that $x^2 w_x + y^2 w_y + z^2 w_z = 0$.*

- Using the definitions of w, u, v together with the chain rule, one finds that

$$\begin{aligned} w_x &= w_u u_x + w_v v_x = -x^{-2} w_u, \\ w_y &= w_u u_y + w_v v_y = y^{-2} w_u - y^{-2} w_v, \\ w_z &= w_u u_z + w_v v_z = z^{-2} w_v. \end{aligned}$$

Once we now combine these three equations, we get the desired identity

$$x^2 w_x + y^2 w_y + z^2 w_z = -w_u + (w_u - w_v) + w_v = 0.$$

6. *Find the minimum value of $f(x,y) = 2x^2 - 4x + 3y^2$ over the closed disk*

$$D = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \leq 9\}.$$

- To find the critical points, we need to solve the equations

$$0 = f_x(x,y) = 4x - 4 = 4(x-1), \qquad 0 = f_y(x,y) = 6y.$$

Thus, the only critical point is $(1,0)$ and this corresponds to the value $f(1,0) = -2$.

- Next, we check the points on the boundary of the disk. Along the boundary,

$$y^2 = 9 - x^2 \quad \Longrightarrow \quad f(x,y) = 2x^2 - 4x + 3(9 - x^2) = -x^2 - 4x + 27$$

and we need to find the minimum value of this function on $[-3, 3]$. Noting that

$$g(x) = -x^2 - 4x + 27 \quad \Longrightarrow \quad g'(x) = -2x - 4 = -2(x+2),$$

we see that the minimum value may only occur at $x = -3$, $x = 3$ or $x = -2$. Since

$$g(-3) = 30, \qquad g(3) = 6, \qquad g(-2) = 31,$$

the smallest value we have found so far is the value $f(1,0) = -2$ we obtained above.

7. *Find the maximum value of* $f(x,y) = x^3 + y^3 - 3xy$ *over the closed triangular region whose vertices are the points* $(0,0)$, $(1,0)$ *and* $(1,2)$.

- To find the critical points, we need to solve the equations

$$0 = f_x(x,y) = 3x^2 - 3y = 3(x^2 - y),$$
$$0 = f_y(x,y) = 3y^2 - 3x = 3(y^2 - x).$$

These give $y = x^2$ and also $x = y^2$, so we easily get

$$x = y^2 = x^4 \quad \Longrightarrow \quad x^4 - x = 0 \quad \Longrightarrow \quad x(x^3 - 1) = 0 \quad \Longrightarrow \quad x = 0, 1.$$

Thus, the only critical points are $(0,0)$ and $(1,1)$, while $f(0,0) = 0$ and $f(1,1) = -1$.

- Next, we check the points on the boundary of the region. Along the horizontal side,

$$y = 0 \quad \Longrightarrow \quad f(x,y) = x^3$$

and we have $0 \leq x \leq 1$, so the maximum value is $f(1,0) = 1$. Along the vertical side,

$$x = 1 \quad \Longrightarrow \quad f(x,y) = y^3 - 3y + 1$$

and we need to find the maximum value of this function on $[0,2]$. Noting that

$$g(y) = y^3 - 3y + 1 \quad \Longrightarrow \quad g'(y) = 3y^2 - 3 = 3(y^2 - 1),$$

we see that the maximum value may only occur at $y = 0$, $y = 2$ or $y = 1$. Since

$$g(0) = 1, \qquad g(2) = 8 - 6 + 1 = 3, \qquad g(1) = 1 - 3 + 1 = -1,$$

the largest value we have found so far is the value $g(2) = 3$ corresponding to $f(1,2) = 3$. It remains to check the boundary points along the hypotenuse. For these points,

$$y = 2x \quad \Longrightarrow \quad f(x,y) = x^3 + (2x)^3 - 3x(2x) = 9x^3 - 6x^2$$

and we need to find the maximum value of this function on $[0,1]$. Noting that

$$h(x) = 9x^3 - 6x^2 \quad \Longrightarrow \quad h'(x) = 27x^2 - 12x = 3x(9x - 4),$$

we see that the maximum value may only occur at $x = 0$, $x = 1$ or $x = 4/9$. Since

$$h(0) = 0, \qquad h(1) = 9 - 6 = 3, \qquad h(4/9) = -32/81,$$

the maximum value over the whole triangular region is the value $f(1,2) = 3$.