MA121 Tutorial Problems #7
Solutions

1. *Letting $\alpha \in \mathbb{R}$ be fixed, find the radius of convergence of the binomial series*

$$f(x) = \sum_{n=0}^{\infty} \frac{\alpha(\alpha-1)(\alpha-2)\cdots(\alpha-n+1)}{n!} \cdot x^n.$$

- One always uses the ratio test to check power series for convergence. In our case,

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{\alpha-n}{n+1} \cdot x \right| = \lim_{n\to\infty} \left| \frac{\alpha/n-1}{1+1/n} \cdot x \right| = |x|$$

so the series converges when $|x| < 1$ and diverges when $|x| > 1$. This also means that the radius of convergence is $R = 1$.

2. *Compute each of the following sums in terms of known functions:*

$$\sum_{n=0}^{\infty} \frac{(-1)^n\, x^{4n+1}}{n!}, \qquad \sum_{n=1}^{\infty} \frac{(-1)^n\, x^{4n+1}}{(2n+1)!}, \qquad \sum_{n=0}^{\infty} \frac{(-1)^n\, x^{6n}}{(2n+2)!}.$$

- Relating the first sum to the Taylor series for the exponential function, we get

$$\sum_{n=0}^{\infty} \frac{(-1)^n\, x^{4n+1}}{n!} = x\sum_{n=0}^{\infty} \frac{(-1)^n\, x^{4n}}{n!} = x\sum_{n=0}^{\infty} \frac{(-x^4)^n}{n!} = xe^{-x^4}.$$

- The second sum is related to the Taylor series for the sine function, namely

$$\sum_{n=1}^{\infty} \frac{(-1)^n\, x^{4n+1}}{(2n+1)!} = \frac{1}{x}\sum_{n=1}^{\infty} \frac{(-1)^n\, (x^2)^{2n+1}}{(2n+1)!} = \frac{\sin(x^2) - x^2}{x}.$$

- To compute the third sum, we shall first shift the index of summation to write

$$\sum_{n=0}^{\infty} \frac{(-1)^n\, x^{6n}}{(2n+2)!} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}\, x^{6n-6}}{(2n)!} = -\frac{1}{x^6}\sum_{n=1}^{\infty} \frac{(-1)^n\, (x^3)^{2n}}{(2n)!}.$$

Relating the last sum to the Taylor series for the cosine function, we now get

$$\sum_{n=0}^{\infty} \frac{(-1)^n\, x^{6n}}{(2n+2)!} = -\frac{1}{x^6} \cdot \Big[\cos(x^3) - 1\Big] = \frac{1-\cos(x^3)}{x^6}.$$

3. *Find the area of the region that lies between the graphs of $f(x) = x + 2$ and $g(x) = x^2$.*

- First of all, let us note that the two graphs intersect when

$$x + 2 = x^2 \quad \Longrightarrow \quad x^2 - x - 2 = 0 \quad \Longrightarrow \quad x = -1, 2.$$

Since a rough sketch of the graphs shows that the graph of f lies above the graph of g between these two points, the desired area is given by

$$\text{Area} = \int_{-1}^{2} [f(x) - g(x)]\, dx = \int_{-1}^{2} \left[x + 2 - x^2\right]\, dx = \left[\frac{x^2}{2} + 2x - \frac{x^3}{3}\right]_{-1}^{2} = \frac{9}{2}.$$

4. *Find the volume of a sphere of radius r.*

- Let R be the region between the graph of $f(x) = \sqrt{r^2 - x^2}$ and the x-axis over $[-r, r]$. Since a sphere of radius r arises by rotating R around the x-axis, its volume is

$$\text{Volume} = \pi \int_{-r}^{r} f(x)^2\, dx = \pi \int_{-r}^{r} (r^2 - x^2)\, dx = \pi \left[r^2 x - \frac{x^3}{3}\right]_{-r}^{r} = \frac{4\pi r^3}{3}.$$

5. *Let R be the region between the graph of $f(x) = \sin x$ and the x-axis over $[0, \pi]$. Find the area of the region R and the volume of the solid obtained by rotating R around the x-axis.*

- Since $\sin x$ is both continuous and non-negative on $[0, \pi]$, the area of the region R is

$$\text{Area} = \int_{0}^{\pi} \sin x\, dx = \Big[-\cos x\Big]_{0}^{\pi} = -\cos\pi + \cos 0 = 2.$$

- Rotating R around the x-axis, one obtains a solid whose volume is given by

$$\begin{aligned} \text{Volume} &= \pi \int_{0}^{\pi} \sin^2 x\, dx = \pi \int_{0}^{\pi} \frac{1 - \cos(2x)}{2}\, dx \\ &= \frac{\pi}{2}\left[x - \frac{\sin(2x)}{2}\right]_{0}^{\pi} = \frac{\pi^2}{2}. \end{aligned}$$