MA121 Tutorial Problems #6
Solutions

1. *Test each of the following series for convergence:*

$$\sum_{n=1}^{\infty} \frac{n+1}{n^2+1}\,, \qquad \sum_{n=1}^{\infty} \frac{n}{2^n}\,, \qquad \sum_{n=2}^{\infty} \frac{\log n}{n}\,, \qquad \sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!}\,.$$

- To test the first series for convergence, we use the limit comparison test with

$$a_n = \frac{n+1}{n^2+1}\,, \qquad b_n = \frac{n}{n^2} = \frac{1}{n}\,.$$

Note that the limit comparison test is, in fact, applicable here because

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n+1}{n^2+1} \cdot n = \lim_{n\to\infty} \frac{n^2+n}{n^2+1} = 1.$$

Since the series $\sum_{n=1}^{\infty} b_n$ is a divergent p-series, the series $\sum_{n=1}^{\infty} a_n$ must also diverge.

- To test the second series for convergence, we use the ratio test. In this case,

$$L = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{n+1}{n} \cdot \frac{2^n}{2^{n+1}} = \lim_{n\to\infty} \frac{n+1}{2n} = \frac{1}{2}$$

is strictly less than 1, so the given series converges by the ratio test.

- When it comes to the third series, the fact that $\log x$ is increasing implies that

$$\sum_{n=2}^{\infty} \frac{\log n}{n} \geq \sum_{n=2}^{\infty} \frac{\log 2}{n}\,.$$

Being bigger than a divergent p-series, the given series must thus diverge by comparison.

- To test the last series for convergence, we use the ratio test. In this case, we have

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)!}{n!} \cdot \frac{(n+1)!}{n!} \cdot \frac{(2n)!}{(2n+2)!} = \frac{(n+1)(n+1)}{(2n+1)(2n+2)} = \frac{n+1}{2(2n+1)}$$

and this implies that

$$L = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{n+1}{4n+2} = \frac{1}{4}\,.$$

Since L is strictly less than 1, the given series must then converge by the ratio test.

2. *Test each of the following series for convergence:*

$$\sum_{n=1}^{\infty} \left(1+\frac{1}{n}\right)^n, \qquad \sum_{n=1}^{\infty} \frac{n+2}{n^3+1}, \qquad \sum_{n=1}^{\infty} \frac{n!}{n^n}, \qquad \sum_{n=1}^{\infty} \log\left(1+\frac{1}{n}\right).$$

- The first series diverges because of the nth term test, namely because

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n = e \neq 0.$$

- To test the second series for convergence, we use the limit comparison test with

$$a_n = \frac{n+2}{n^3+1}, \qquad b_n = \frac{n}{n^3} = \frac{1}{n^2}.$$

  Note that the limit comparison test is, in fact, applicable here because

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n+2}{n^3+1} \cdot n^2 = \lim_{n\to\infty} \frac{n^3+2n^2}{n^3+1} = 1.$$

  Since the series $\sum_{n=1}^{\infty} b_n$ is a convergent p-series, the series $\sum_{n=1}^{\infty} a_n$ must also converge.

- To test the third series for convergence, we use the ratio test. In this case, we have

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)!}{n!} \cdot \frac{n^n}{(n+1)^{n+1}} = \frac{n^n}{(n+1)^n} = \left(\frac{n}{n+1}\right)^n$$

  and this implies that

$$L = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^n = \frac{1}{e}.$$

  Since $e > 1$, this limit is strictly less than 1 and the given series converges.

- To test the last series for convergence, we use the limit comparison test with

$$a_n = \log\left(1+\frac{1}{n}\right), \qquad b_n = \frac{1}{n}.$$

  Note that the limit comparison test is, in fact, applicable here because

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} n \log\left(1+\frac{1}{n}\right) = \lim_{n\to\infty} \log\left(1+\frac{1}{n}\right)^n = \log e = 1.$$

  Since the series $\sum_{n=1}^{\infty} b_n$ is a divergent p-series, the series $\sum_{n=1}^{\infty} a_n$ must also diverge.

3. *Compute each of the following sums:*

$$\sum_{n=0}^{\infty} \frac{3^{n+2}}{2^{2n+1}}, \qquad \sum_{n=1}^{\infty} \frac{5^{n+1}}{2^{3n}}.$$

- When it comes to the first sum, the formula for a geometric series gives

$$\sum_{n=0}^{\infty} \frac{3^{n+2}}{2^{2n+1}} = \frac{9}{2} \cdot \sum_{n=0}^{\infty} \frac{3^n}{4^n} = \frac{9}{2} \cdot \frac{1}{1-3/4} = \frac{9}{2} \cdot 4 = 18.$$

- When it comes to the second sum, a similar computation gives

$$\sum_{n=1}^{\infty} \frac{5^{n+1}}{2^{3n}} = 5 \cdot \sum_{n=1}^{\infty} \frac{5^n}{8^n} = 5 \cdot \left[\frac{1}{1-5/8} - 1 \right] = 5 \cdot \frac{5}{3} = \frac{25}{3}.$$