MA121 Tutorial Problems #5
Solutions

1. *Compute each of the following integrals:*

$$\int \log(1+x^2)\,dx, \qquad \int \frac{x^3-x+1}{x+2}\,dx, \qquad \int \frac{dx}{1+e^x}, \qquad \int \frac{x^2+1}{x^3+x^2}\,dx.$$

- To compute the first integral, we integrate by parts to find that

$$\int \log(1+x^2)\,dx = \int x' \log(1+x^2)\,dx = x\log(1+x^2) - \int \frac{2x^2}{1+x^2}\,dx.$$

Using division of polynomials, one can easily check that

$$\frac{2x^2}{1+x^2} = 2 - \frac{2}{1+x^2} \quad \Longrightarrow \quad \int \frac{2x^2}{1+x^2}\,dx = 2x - 2\arctan x + C.$$

Once we now combine the last two equations, we get

$$\int \log(1+x^2)\,dx = x\log(1+x^2) - 2x + 2\arctan x + C.$$

- To compute the second integral, we use division of polynomials to write

$$\frac{x^3-x+1}{x+2} = x^2 - 2x + 3 - \frac{5}{x+2}\,.$$

Integrating this equation term by term, we then easily deduce that

$$\int \frac{x^3-x+1}{x+2}\,dx = \frac{x^3}{3} - x^2 + 3x - 5\log|x+2| + C.$$

- For the third integral, we set $u = e^x$. This gives $du = e^x\,dx$, hence also

$$\int \frac{dx}{1+e^x} = \int \frac{e^x\,dx}{e^x(1+e^x)} = \int \frac{du}{u(1+u)}\,.$$

Focusing on the rightmost expression, we now use partial fractions to write

$$\frac{1}{u(1+u)} = \frac{A}{u} + \frac{B}{1+u} \tag{1}$$

for some constants A, B that need to be determined. Clearing denominators gives

$$1 = A(1+u) + Bu$$

and we can look at some suitable choices of u to find that

$$u = 0, \quad u = -1 \qquad \Longrightarrow \qquad 1 = A, \quad 1 = -B.$$

Returning to equation (1), we conclude that

$$\int \frac{du}{u(1+u)} = \int \frac{1}{u}\, du - \int \frac{1}{1+u}\, du = \log|u| - \log|1+u| + C.$$

Since $u = e^x$ by above, this also implies that

$$\int \frac{dx}{1+e^x} = \int \frac{du}{u(1+u)} = x - \log(1+e^x) + C.$$

- To compute the fourth integral, we factor the denominator and we write

$$\frac{x^2+1}{x^3+x^2} = \frac{x^2+1}{x^2(x+1)} = \frac{Ax+B}{x^2} + \frac{C}{x+1} \tag{2}$$

for some constants A, B, C that need to be determined. Clearing denominators gives

$$x^2 + 1 = (Ax+B)(x+1) + Cx^2$$

and we can now look at some suitable choices of x to find

$$x = 0, \quad x = -1 \qquad \Longrightarrow \qquad 1 = B, \quad 2 = C.$$

Taking $x = 1$ as our final choice, we also get

$$2 = 2(A+B) + C \quad \Longrightarrow \quad A = \frac{2 - 2B - C}{2} = \frac{2-2-2}{2} = -1.$$

Returning to equation (2), we may finally conclude that

$$\begin{aligned}\int \frac{x^2+1}{x^3+x^2}\, dx &= -\int \frac{1}{x}\, dx + \int \frac{1}{x^2}\, dx + \int \frac{2}{x+1}\, dx \\ &= -\log|x| - x^{-1} + 2\log|x+1| + C.\end{aligned}$$

2. *Compute each of the following integrals:*

$$\int \frac{2x+3}{x^2-9}\,dx, \qquad \int \frac{x+7}{x^2(x+2)}\,dx, \qquad \int \frac{2\sin x\cdot\cos x}{\sin^2 x-4}\,dx, \qquad \int x\cdot\arctan x\,dx.$$

- To compute the first integral, we use partial fractions to write

$$\frac{2x+3}{x^2-9} = \frac{2x+3}{(x-3)(x+3)} = \frac{A}{x-3} + \frac{B}{x+3} \tag{3}$$

for some constants A, B that need to be determined. Clearing denominators gives

$$2x+3 = A(x+3) + B(x-3)$$

and we can now look at some suitable choices of x. We set $x=3$ to find

$$2\cdot 3+3 = 6A \quad\Longrightarrow\quad 9 = 6A \quad\Longrightarrow\quad A = 9/6 = 3/2$$

and then we set $x=-3$ to find

$$-2\cdot 3+3 = -6B \quad\Longrightarrow\quad -3 = -6B \quad\Longrightarrow\quad B = 3/6 = 1/2.$$

Returning to equation (3), we may finally conclude that

$$\begin{aligned}\int \frac{2x+3}{x^2-9}\,dx &= \int \frac{3/2}{x-3}\,dx + \int \frac{1/2}{x+3}\,dx \\ &= \frac{3}{2}\cdot\log|x-3| + \frac{1}{2}\cdot\log|x+3| + C.\end{aligned}$$

- To compute the second integral, we use partial fractions to write

$$\frac{x+7}{x^2(x+2)} = \frac{Ax+B}{x^2} + \frac{C}{x+2}\,. \tag{4}$$

Clearing denominators, as before, we find that

$$x+7 = (Ax+B)(x+2) + Cx^2.$$

We now set $x=0$ in the last equation to get

$$7 = 2B \quad\Longrightarrow\quad B = 7/2,$$

we set $x=-2$ to similarly get

$$5 = 4C \quad\Longrightarrow\quad C = 5/4,$$

and we set $x=-1$ to get

$$6 = -A+B+C \quad\Longrightarrow\quad A = B+C-6 = \frac{7}{2}+\frac{5}{4}-6 = -\frac{5}{4}\,.$$

Returning to equation (4), we may finally conclude that

$$\begin{aligned}\int \frac{x+7}{x^2(x+2)}\,dx &= -\int \frac{5/4}{x}\,dx + \int \frac{7/2}{x^2}\,dx + \int \frac{5/4}{x+2}\,dx \\ &= -\frac{5}{4}\cdot \log|x| - \frac{7}{2x} + \frac{5}{4}\cdot \log|x+2| + C.\end{aligned}$$

- Setting $u = \sin^2 x - 4$, we get $du = 2\sin x\cos x\,dx$, hence also

$$\int \frac{2\sin x\cos x}{\sin^2 x - 4}\,dx = \int \frac{du}{u} = \log|u| + C.$$

Note that u is negative here because $u = \sin^2 x - 4 \leq 1 - 4 = -3$. This means that

$$\int \frac{2\sin x\cos x}{\sin^2 x - 4}\,dx = \log(-u) + C = \log(4 - \sin^2 x) + C.$$

- First of all, we integrate by parts to find that

$$\int x\arctan x\,dx = \int \left(\frac{x^2}{2}\right)' \arctan x\,dx = \frac{x^2\arctan x}{2} - \frac{1}{2}\int \frac{x^2}{1+x^2}\,dx.$$

Using division of polynomials, one can easily check that

$$\frac{x^2}{1+x^2} = 1 - \frac{1}{1+x^2} \quad\Longrightarrow\quad \int \frac{x^2}{1+x^2}\,dx = x - \arctan x + C.$$

Once we now combine the last two equations, we get

$$\int x\arctan x\,dx = \frac{x^2\arctan x}{2} - \frac{x}{2} + \frac{\arctan x}{2} + C.$$