MA121 Tutorial Problems #4
Solutions

1. *Show that* $\sin^2 x + \cos^2 x = 1$ *for all* $x \in \mathbb{R}$.

Letting $f(x) = \sin^2 x + \cos^2 x$ for convenience, one easily finds that

$$f'(x) = 2\sin x(\sin x)' + 2\cos x(\cos x)' = 2\sin x\cos x - 2\cos x\sin x = 0.$$

This shows that $f(x)$ is actually constant, hence $f(x) = f(0) = \sin^2 0 + \cos^2 0 = 1$.

2. *Let* f *be a non-negative function which is integrable on* $[0,1]$ *with* $f(x) = 0$ *for all* $x \in \mathbb{Q}$. *Show that* $\int_0^1 f(x)\,dx = 0$.

Suppose that $P = \{x_0, x_1, \ldots, x_n\}$ is a partition of $[0,1]$. Then we must clearly have

$$\inf_{[x_k, x_{k+1}]} f(x) = 0 \quad \text{for each } 0 \leq k \leq n-1$$

because f is non-negative and since every subinterval contains a rational. Thus,

$$S^-(f,P) = \sum_{k=0}^{n-1} \inf_{[x_k, x_{k+1}]} f(x) \cdot (x_{k+1} - x_k) = 0$$

as well. Taking the supremum of both sides, we conclude that

$$\int_0^1 f(x)\,dx = \sup_P \{S^-(f,P)\} = \sup_P \{0\} = 0.$$

3. *Suppose* f *is continuous on* $[a,b]$. *Show that there exists some* $c \in (a,b)$ *such that*

$$\int_a^b f(t)\,dt = (b-a) \cdot f(c).$$

As a hint, apply the mean value theorem to the function $F(x) = \int_a^x f(t)\,dt$.

According to the mean value theorem, there exists some $c \in (a,b)$ such that

$$\frac{F(b) - F(a)}{b - a} = F'(c).$$

In addition, we have $F'(x) = f(x)$ for all x, and we also have

$$F(a) = \int_a^a f(t)\,dt = 0, \qquad F(b) = \int_a^b f(t)\,dt.$$

Once we now combine all these facts, we may conclude that

$$F(b) - F(a) = (b-a) \cdot F'(c) \quad \Longrightarrow \quad \int_a^b f(t)\,dt = (b-a) \cdot f(c).$$

4. *Compute each of the following integrals:*

$$\int \frac{\sin(1/x)}{x^2}\,dx, \qquad \int (x+1)(x+2)^5\,dx, \qquad \int \frac{x}{\sqrt{x+1}}\,dx, \qquad \int xe^x\,dx.$$

- For the first integral, the substitution $u = 1/x = x^{-1}$ gives $du = -x^{-2}\,dx$ so that

$$\int \frac{\sin(1/x)}{x^2}\,dx = -\int \sin u\,du = \cos u + C = \cos(1/x) + C.$$

- For the second integral, set $u = x+2$. This gives $du = dx$ and $x+1 = u-1$, hence

$$\begin{aligned}\int (x+1)(x+2)^5\,dx &= \int (u-1)\,u^5\,du = \int (u^6 - u^5)\,du \\ &= \frac{u^7}{7} - \frac{u^6}{6} + C = \frac{(x+2)^7}{7} - \frac{(x+2)^6}{6} + C.\end{aligned}$$

- For the third integral, set $u = x+1$. This gives $du = dx$ and $x = u-1$ so that

$$\begin{aligned}\int \frac{x}{\sqrt{x+1}}\,dx &= \int \frac{u-1}{\sqrt{u}}\,du = \int \frac{u-1}{u^{1/2}}\,du \\ &= \int \left(u^{1/2} - u^{-1/2}\right)du = \frac{2u^{3/2}}{3} - 2u^{1/2} + C \\ &= \frac{2(x+1)^{3/2}}{3} - 2(x+1)^{1/2} + C.\end{aligned}$$

- For the last integral, one may integrate by parts to find that

$$\int xe^x\,dx = \int x\,(e^x)'\,dx = xe^x - \int e^x\,dx = xe^x - e^x + C.$$

An alternative way of getting this answer is by using the tabular integration below.

Differentiating	Integrating
x	e^x
1	e^x
0	e^x

5. *Compute each of the following integrals:*

$$\int \sin^3 x\, dx, \qquad \int \frac{x}{e^x}\, dx, \qquad \int e^{\sqrt{x}}\, dx, \qquad \int \frac{\log x}{x^2}\, dx.$$

- To compute the first integral, it is convenient to write it in the form

$$\int \sin^3 x\, dx = \int (1-\cos^2 x)\sin x\, dx = \int \sin x\, dx - \int \cos^2 x \sin x\, dx.$$

Using the substitution $u = \cos x$, we then get $du = -\sin x\, dx$, hence also

$$\int \sin^3 x\, dx = -\cos x + \int u^2\, du = -\cos x + \frac{u^3}{3} + C = -\cos x + \frac{\cos^3 x}{3} + C.$$

- For the second integral, one may integrate by parts to find that

$$\int xe^{-x}\, dx = \int x\left(-e^{-x}\right)'\, dx = -xe^{-x} + \int e^{-x}\, dx = -xe^{-x} - e^{-x} + C.$$

An alternative way of getting this answer is by using the tabular integration below.

Differentiating	Integrating
x	e^{-x}
1	$-e^{-x}$
0	e^{-x}

- For the third integral, we set $u = \sqrt{x}$. This gives $x = u^2$ so that $dx = 2u\, du$ and

$$\int e^{\sqrt{x}}\, dx = 2\int ue^u\, du.$$

Using our computation from the previous problem, we may thus conclude that

$$\int e^{\sqrt{x}}\, dx = 2ue^u - 2e^u + C = 2\sqrt{x}\, e^{\sqrt{x}} - 2e^{\sqrt{x}} + C.$$

- To compute the last integral, we integrate by parts to find that

$$\begin{aligned}\int x^{-2}\cdot \log x\, dx &= \int \left(-x^{-1}\right)'\cdot \log x\, dx = -x^{-1}\cdot \log x + \int x^{-1}\cdot (\log x)'\, dx \\ &= -x^{-1}\log x + \int x^{-2}\, dx = -x^{-1}\log x - x^{-1} + C.\end{aligned}$$