MA121 Tutorial Problems #3
Solutions

1. *Let f be the function defined by*

$$f(x) = \left\{ \begin{array}{cc} \frac{8x^3+4x-3}{2x-1} & \text{if } x \neq 1/2 \\ 5 & \text{if } x = 1/2 \end{array} \right\}.$$

Show that f is continuous at all points.

Since f agrees with a rational function on the open interval $(-\infty, 1/2)$, it is continuous on that interval by a result of ours. Similarly, f is continuous on $(1/2, +\infty)$ as well, so it remains to check continuity at $y = 1/2$. In other words, it remains to check that

$$\lim_{x \to 1/2} f(x) = f(1/2).$$

Using division of polynomials to evaluate the limit, one now finds that

$$\lim_{x \to 1/2} f(x) = \lim_{x \to 1/2} \frac{8x^3 + 4x - 3}{2x - 1} = \lim_{x \to 1/2} (4x^2 + 2x + 3).$$

Since limits of polynomials can be computed by simple substitution, this also implies

$$\lim_{x \to 1/2} f(x) = \lim_{x \to 1/2} (4x^2 + 2x + 3) = 4 \cdot \frac{1}{4} + 2 \cdot \frac{1}{2} + 3 = 5 = f(1/2).$$

2. *Evaluate each of the following limits:*

$$\lim_{x \to +\infty} \frac{6x^2 - 5}{2 - 3x^2}, \qquad \lim_{x \to -\infty} \frac{6x^3 - 5x^2 + 2}{1 - 3x + x^4}.$$

To compute the limit of a rational function as $x \to \pm\infty$, one divides both the numerator and the denominator by the highest power of x in the denominator. In our case,

$$\lim_{x \to +\infty} \frac{6x^2 - 5}{2 - 3x^2} = \lim_{x \to +\infty} \frac{6 - 5/x^2}{2/x^2 - 3} = \frac{6 - 0}{0 - 3} = -2$$

and a similar computation gives

$$\lim_{x \to -\infty} \frac{6x^3 - 5x^2 + 2}{1 - 3x + x^4} = \lim_{x \to -\infty} \frac{6/x - 5/x^2 + 2/x^4}{1/x^4 - 3/x^3 + 1} = \frac{0 - 0 + 0}{0 - 0 + 1} = 0.$$

3. *Find the maximum value of $f(x) = 3x^4 - 16x^3 + 18x^2$ over the closed interval $[-1, 2]$.*

Since we are dealing with a closed interval, it suffices to check the endpoints, the points at which f' does not exist and the points at which f' is equal to zero. In our case,

$$f'(x) = 12x^3 - 48x^2 + 36x = 12x(x^2 - 4x + 3) = 12x(x - 1)(x - 3)$$

and the only points at which the maximum value may occur are

$$x = -1, \qquad x = 2, \qquad x = 0, \qquad x = 1, \qquad x = 3.$$

We exclude the rightmost point, as this fails to lie in $[-1, 2]$, and we now compute

$$f(-1) = 37, \qquad f(2) = -8, \qquad f(0) = 0, \qquad f(1) = 5.$$

Based on these facts, we may finally conclude that the maximum value is $f(-1) = 37$.

4. *Find the minimum value of* $f(x) = (2x^2 - 5x + 2)^3$ *over the closed interval* $[0, 1]$.

Since we are dealing with a closed interval, it suffices to check the endpoints, the points at which f' does not exist and the points at which f' is equal to zero. In our case,

$$\begin{aligned} f'(x) &= 3(2x^2 - 5x + 2)^2 \cdot (2x^2 - 5x + 2)' \\ &= 3(2x^2 - 5x + 2)^2 \cdot (4x - 5) \end{aligned}$$

is zero when $x = 5/4$ and also when the quadratic factor is zero, namely when

$$x = \frac{5 \pm \sqrt{25 - 4 \cdot 2 \cdot 2}}{2 \cdot 2} = \frac{5 \pm 3}{4} \implies x = 2, \quad x = 1/2.$$

Since $x = 5/4$ and $x = 2$ do not lie in the given closed interval, this means that

$$x = 0, \qquad x = 1, \qquad x = 1/2$$

are the only points at which the minimum value may occur. Once we now compute

$$f(0) = 8, \qquad f(1) = -1, \qquad f(1/2) = 0,$$

we may finally conclude that the minimum value is $f(1) = -1$.

5. *Find the values of* x *for which* $f'(x) = 0$ *in each of the following cases:*

$$f(x) = \frac{x^2}{1 + x^2}, \qquad f(x) = x(x^2 - 9)^4.$$

• When it comes to the first function, an application of the quotient rule gives

$$f'(x) = \frac{2x \cdot (1 + x^2) - 2x \cdot x^2}{(1 + x^2)^2} = \frac{2x}{(1 + x^2)^2}$$

and this is zero if and only if $x = 0$.

• When it comes to the second function, we have

$$f'(x) = 1 \cdot (x^2 - 9)^4 + x \cdot 4(x^2 - 9)^3 \cdot (x^2 - 9)'$$

by the product and the chain rule. We simplify this expression and factor to get

$$\begin{aligned} f'(x) &= (x^2 - 9)^4 + 4x(x^2 - 9)^3 \cdot 2x = (x^2 - 9)^3 \cdot (x^2 - 9 + 8x^2) \\ &= (x^2 - 9)^3 \cdot 9(x^2 - 1). \end{aligned}$$

Based on this factorization, it is clear that $f'(x) = 0$ when either $x = \pm 3$ or $x = \pm 1$.

6. *Show that the polynomial* $f(x) = x^3 - 3x + 1$ *has three roots in the interval* $(-2, 2)$*. As a hint, you might wish to compute the values of* f *at the points* ± 2, ± 1 *and* 0.

Being a polynomial, f is continuous on the closed interval $[-2, -1]$ and we also have

$$f(-2) = -1 < 0, \qquad f(-1) = 3 > 0.$$

Thus, f must have a root in $(-2, -1)$ by Bolzano's theorem. Using the facts that

$$f(0) = 1 > 0, \qquad f(1) = -1 < 0, \qquad f(2) = 3 > 0,$$

we similarly find that another root exists in $(0, 1)$ and that a third root exists in $(1, 2)$. In particular, f has three roots in $(-2, 2)$, as needed.

7. *Show that the polynomial* $f(x) = x^3 - 4x^2 - 3x + 1$ *has exactly one root in* $[0, 2]$.

Being a polynomial, f is continuous on the closed interval $[0, 2]$ and we also have

$$f(0) = 1 > 0, \qquad f(2) = -13 < 0.$$

Thus, f has a root in $(0, 2)$ by Bolzano's theorem and this root certainly lies in $[0, 2]$ as well. Suppose now that f has two roots in $[0, 2]$. By Rolle's theorem, f' must then have a root in $[0, 2]$ as well. On the other hand, the roots of $f'(x) = 3x^2 - 8x - 3$ are

$$x = \frac{8 \pm \sqrt{64 + 4 \cdot 3 \cdot 3}}{2 \cdot 3} = \frac{8 \pm 10}{6} \quad \Longrightarrow \quad x = 3, \quad x = -1/3.$$

Since neither of those lies in $[0, 2]$, we conclude that f cannot have two roots in $[0, 2]$.