MA121 Tutorial Problems #2
Solutions

1. *Show that there exists some* $0 < x < 1$ *such that* $4x^3 + 3x = 2x^2 + 2$.

Let $f(x) = 4x^3 + 3x - 2x^2 - 2$ for all $x \in [0, 1]$. Being a polynomial, f is then continuous on the closed interval $[0, 1]$. Once we now note that

$$f(0) = -2 < 0, \qquad f(1) = 4 + 3 - 2 - 2 = 3 > 0,$$

we may use Bolzano's theorem to conclude that $f(x) = 0$ for some $x \in (0, 1)$. This also implies that $4x^3 + 3x = 2x^2 + 2$ for some $0 < x < 1$, as needed.

2. *Evaluate each of the following limits:*

$$\lim_{x\to 1} \frac{6x^3 - 5x^2 - 3x + 2}{x + 1}, \qquad \lim_{x\to 1} \frac{6x^3 - 5x^2 - 3x + 2}{x - 1}.$$

• When it comes to the first limit, one easily finds that

$$\lim_{x\to 1} \frac{6x^3 - 5x^2 - 3x + 2}{x + 1} = \frac{6 - 5 - 3 + 2}{1 + 1} = \frac{0}{2} = 0$$

because limits of rational functions can be computed by simple substitution.

• When it comes to the second limit, division of polynomials gives

$$\lim_{x\to 1} \frac{6x^3 - 5x^2 - 3x + 2}{x - 1} = \lim_{x\to 1} (6x^2 + x - 2) = 6 + 1 - 2 = 5$$

because $x \neq 1$ and since limits of polynomials can be computed by simple substitution.

3. *Determine the values of* x *for which* $6x^2 < 7x - 2$.

We need to determine the values of x for which $f(x) = 6x^2 - 7x + 2$ is negative. Note that the two roots of this quadratic are given by

$$x = \frac{7 \pm \sqrt{7^2 - 4 \cdot 6 \cdot 2}}{2 \cdot 6} = \frac{7 \pm 1}{12} \quad \Longrightarrow \quad x = 1/2, \quad x = 2/3.$$

This gives the factorization $f(x) = 6(x - 1/2)(x - 2/3)$ and the sign of $f(x)$ is now easy to determine. In view of the table below, we have $f(x) < 0$ if and only if $x \in (1/2, 2/3)$.

x		$1/2$		$2/3$	
$x - 1/2$	$-$		$+$		$+$
$x - 2/3$	$-$		$-$		$+$
$f(x)$	$+$		$-$		$+$

4. *Let f be the function defined by*

$$f(x) = \left\{ \begin{array}{cc} 2x & \text{if } x \in \mathbb{Q} \\ 5 - 3x & \text{if } x \notin \mathbb{Q} \end{array} \right\}.$$

Show that f is continuous at $y = 1$.

In this case, we have

$$|f(x) - f(1)| = |f(x) - 2| = \left\{ \begin{array}{cc} 2|x-1| & \text{if } x \in \mathbb{Q} \\ 3|1-x| & \text{if } x \notin \mathbb{Q} \end{array} \right\}.$$

Given any $\varepsilon > 0$, we can then set $\delta = \varepsilon/3$ to find that

$$|x - 1| < \delta \quad \Longrightarrow \quad |f(x) - f(1)| \leq 3|x - 1| < 3\delta = \varepsilon.$$

5. *Suppose that f is continuous on $[0, 1]$ and that $0 < f(x) < 1$ for all $x \in [0, 1]$. Show that there exists some $0 < c < 1$ such that $f(c) = c$.*

Let $g(x) = f(x) - x$ for all $x \in [0, 1]$. Being the difference of two continuous functions, g is then continuous on the closed interval $[0, 1]$. Once we now note that

$$g(0) = f(0) > 0, \qquad g(1) = f(1) - 1 < 0,$$

we may use Bolzano's theorem to conclude that $g(c) = 0$ for some $c \in (0, 1)$. This also implies that $f(c) = c$ for some $0 < c < 1$, as needed.

6. *Determine the values of x for which $x^3 < 9x$.*

We need to determine the values of x for which

$$f(x) = x^3 - 9x = x(x^2 - 9) = x(x - 3)(x + 3)$$

is negative. By the table below, this is the case when either $x < -3$ or else $0 < x < 3$.

x		-3	0	3
x	$-$	$-$	$+$	$+$
$x - 3$	$-$	$-$	$-$	$+$
$x + 3$	$-$	$+$	$+$	$+$
$f(x)$	$-$	$+$	$-$	$+$

7. *Let f be the function defined by*

$$f(x) = \left\{ \begin{array}{cc} 3x & \text{if } x \leq 1 \\ 4x - 1 & \text{if } x > 1 \end{array} \right\}.$$

Show that f is continuous at all points.

• Since f agrees with a polynomial on the open interval $(-\infty, 1)$ and since polynomials are known to be continuous, it is clear that f is continuous on $(-\infty, 1)$. Using the exact same argument, we find that f is continuous on the open interval $(1, +\infty)$ as well.

• To check continuity at the remaining point $y = 1$, let us first note that

$$|f(x) - f(1)| = |f(x) - 3| = \left\{ \begin{array}{ll} 3|x-1| & \text{if } x \leq 1 \\ 4|x-1| & \text{if } x > 1 \end{array} \right\}.$$

Given any $\varepsilon > 0$, we can then set $\delta = \varepsilon/4$ to find that

$$|x - 1| < \delta \quad \Longrightarrow \quad |f(x) - f(1)| \leq 4|x - 1| < 4\delta = \varepsilon.$$

This establishes continuity at $y = 1$ as well, so f is continuous at all points.

8. *Show that the function f defined by*

$$f(x) = \left\{ \begin{array}{ll} 2x & \textit{if } x \leq 1 \\ x+2 & \textit{if } x > 1 \end{array} \right\}$$

is discontinuous at $y = 1$.

We will show that the ε-δ definition of continuity fails when $\varepsilon = 1$. Suppose it does not fail. Since $f(1) = 2$, there must then exist some $\delta > 0$ such that

$$|x - 1| < \delta \quad \Longrightarrow \quad |f(x) - 2| < 1. \tag{$*$}$$

Let us now examine the last equation for the choice $x = 1 + \frac{\delta}{2}$. On one hand, we have

$$|x - 1| = \frac{\delta}{2} < \delta,$$

so the assumption in equation $(*)$ holds. On the other hand, we also have

$$|f(x) - 2| = |x + 2 - 2| = 1 + \frac{\delta}{2} > 1$$

because $x = 1 + \frac{\delta}{2} > 1$ here. This actually violates the conclusion in equation $(*)$.