MA121 Tutorial Problems #1
Solutions

1. *Make a table listing the min, inf, max and sup of each of the following sets; write DNE for all quantities which fail to exist. You need not justify any of your answers.*

(a) $A = \left\{ n \in \mathbb{N} : \frac{1}{n} > \frac{1}{3} \right\}$

(b) $B = \{ x \in \mathbb{R} : x > 1 \text{ and } 2x \leq 5 \}$

(c) $C = \{ x \in \mathbb{Z} : x > 1 \text{ and } 2x \leq 5 \}$

(d) $D = \{ x \in \mathbb{R} : x < y \text{ for all } y > 0 \}$

(e) $E = \{ x \in \mathbb{R} : x > y \text{ for all } y > 0 \}$

- A complete list of answers is provided by the following table.

	min	inf	max	sup
A	1	1	2	2
B	DNE	1	5/2	5/2
C	2	2	2	2
D	DNE	DNE	0	0
E	DNE	DNE	DNE	DNE

- The set A contains all $n \in \mathbb{N}$ with $n < 3$; this means that $A = \{1, 2\}$.
- The set B contains all $x \in \mathbb{R}$ with $1 < x \leq 5/2$; this means that $B = (1, 5/2]$.
- The set C contains all integers x with $1 < x \leq 5/2$; this means that $C = \{2\}$.
- The set D contains the real numbers x which are smaller than all positive reals; this means that $D = (-\infty, 0]$.
- The set E contains the real numbers x which are bigger than all positive reals; as you can easily convince yourselves, there are no such real numbers, hence E is empty.

2. *Let $x \in \mathbb{R}$ be such that $x > -1$. Show that $(1+x)^n \geq 1 + nx$ for all $n \in \mathbb{N}$.*

- We use induction to prove the given inequality for all $n \in \mathbb{N}$.
- When $n = 1$, the given inequality holds because $(1+x)^1 = 1 + x = 1 + nx$.
- Suppose that the inequality holds for some n, in which case

$$(1+x)^n \geq 1 + nx.$$

Since $1 + x > 0$ by assumption, we may then multiply this inequality by $1 + x$ to get

$$(1+x)^{n+1} \geq (1+nx)(1+x) = 1 + (n+1)x + nx^2 \geq 1 + (n+1)x$$

because $nx^2 \geq 0$. This actually proves the given inequality for $n + 1$, as needed.

3. *Let $f(x) = x^2 - 4x$ for all $x \in \mathbb{R}$. Show that* $\inf f(x) = -4$, *whereas* $\inf_{0 \le x \le 1} f(x) = -3$.

- To prove the first statement, it suffices to show that $\min f(x) = -4$. Once a minimum is known to exist, that is, the infimum also does and the two are equal. Note that

$$f(x) + 4 = x^2 - 4x + 4 = (x-2)^2 \ge 0$$

and that equality holds in the last inequality when $x = 2$. In particular, $f(x) \ge -4$ for all $x \in \mathbb{R}$ and we also have $f(x) = -4$ when $x = 2$, hence $\min f(x) = -4$.

- The proof of the second statement is quite similar. In this case, one notes that

$$f(x) + 3 = x^2 - 4x + 3 = (x-1)(x-3) \ge 0$$

for all $0 \le x \le 1$ and that equality holds in the above inequality when $x = 1$. Based on these facts, we have $\min_{0 \le x \le 1} f(x) = -3$, however this also implies $\inf_{0 \le x \le 1} f(x) = -3$.

4. *Let A, B be nonempty subsets of $\mathbb{R}$ such that* $\sup A < \sup B$. *Show that there exists an element $b \in B$ which is an upper bound of A.*

Since $\sup A$ is smaller than the least upper bound of B, we see that $\sup A$ cannot be an upper bound of B. This means that some element $b \in B$ is such that $b > \sup A$. Using the fact that $\sup A$ is an upper bound of A, we now get $b > \sup A \ge a$ for all $a \in A$. This means that b itself is an upper bound of A.

5. *Given any real number $x \neq 1$, show that*

$$1 + x + \ldots + x^n = \frac{1 - x^{n+1}}{1 - x} \quad \textit{for all } n \in \mathbb{N}.$$

- We use induction to establish the given identity for all $n \in \mathbb{N}$.
- When $n = 1$, we can use division of polynomials to find that

$$\frac{1 - x^{n+1}}{1 - x} = \frac{1 - x^2}{1 - x} = 1 + x = 1 + x^1$$

because $x \neq 1$ by assumption. This proves the given identity for the case $n = 1$.

- Suppose the identity holds for some n, in which case

$$1 + x + \ldots + x^n = \frac{1 - x^{n+1}}{1 - x}.$$

Adding x^{n+1} to both sides, we then get

$$1 + x + \ldots + x^{n+1} = \frac{1 - x^{n+1}}{1 - x} + x^{n+1} = \frac{1 - x^{n+1} + x^{n+1} - x^{n+2}}{1 - x}.$$

Simplifying the rightmost expression, we finally arrive at

$$1 + x + \ldots + x^{n+1} = \frac{1 - x^{n+2}}{1 - x} = \frac{1 - x^{n+1+1}}{1 - x}.$$

Since this proves the given identity for $n + 1$, the identity holds for all $n \in \mathbb{N}$, indeed.