MA121 Tutorial Problems #8

1. Letting $f(x, y, z) = x^2yz$, find the rate at which f is changing at the point $(1, 2, 1)$ in the direction of the vector $\mathbf{v} = \langle 1, 2, 2\rangle$.

2. Letting $f(x, y) = x^2e^y + xy$, find the direction in which f increases most rapidly at the point $(2, 0)$. What is the exact rate of change in that direction?

3. Suppose that $u = x^2 - y^2$, where $x = r\cos\theta$ and $y = r\sin\theta$. Compute u_r and u_θ.

4. Suppose that $z = z(r, s, t)$, where $r = u - v$, $s = v - w$ and $t = w - u$. Assuming that all partial derivatives exist, show that $z_u + z_v + z_w = 0$.

5. Suppose that $w = w(u, v)$, where $u = x^{-1} - y^{-1}$ and $v = y^{-1} - z^{-1}$. Assuming that all partial derivatives exist, show that $x^2w_x + y^2w_y + z^2w_z = 0$.

6. Find the minimum value of $f(x, y) = 2x^2 - 4x + 3y^2$ over the closed disk

$$D = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 9\}.$$

7. Find the maximum value of $f(x, y) = x^3 + y^3 - 3xy$ over the closed triangular region whose vertices are the points $(0, 0)$, $(1, 0)$ and $(1, 2)$.

- These are all practice problems, not a homework assignment.
- However, part of your next homework assignment will be based on these problems.
- In case you get stuck, some hints are provided on the other page of this sheet.

Hints and comments

1. To find a unit vector $\mathbf{u}$ in the direction of $\mathbf{v}$, we need to divide $\mathbf{v}$ by its length, namely

$$||\mathbf{v}|| = \sqrt{1^2 + 2^2 + 2^2} = 3 \quad \Longrightarrow \quad \mathbf{u} = \frac{1}{3}\mathbf{v} = \langle 1/3, 2/3, 2/3 \rangle.$$

The desired rate of change is given by the directional derivative $D_{\mathbf{u}} f = \nabla f \cdot \mathbf{u}$. Since

$$\nabla f(x, y, z) = \langle 2xyz, x^2 z, x^2 y \rangle \quad \Longrightarrow \quad \nabla f(1, 2, 1) = \langle 4, 1, 2 \rangle,$$

we may thus conclude that the desired rate of change is $D_{\mathbf{u}} f = \nabla f \cdot \mathbf{u} = 10/3$.

2. The direction of most rapid increase is that of the gradient vector

$$\nabla f(x, y) = \langle f_x, f_y \rangle = \langle 2xe^y + y, x^2 e^y + x \rangle \quad \Longrightarrow \quad \nabla f(2, 0) = \langle 4, 6 \rangle.$$

Find a unit vector $\mathbf{u}$ in the same direction and compute $D_{\mathbf{u}} f = \nabla f \cdot \mathbf{u} = 2\sqrt{13}$.

3. We have $u_r = u_x x_r + u_y y_r = 2x \cos\theta - 2y \sin\theta$ and similarly $u_\theta = -2xr \sin\theta - 2yr \cos\theta$.

4. In this case, $z_u = z_r r_u + z_s s_u + z_t t_u$ and you can simplify this expression to get

$$z_u = z_r r_u + z_s s_u + z_t t_u = z_r - z_t.$$

Argue similarly to find that $z_v = -z_r + z_s$ and $z_w = -z_s + z_t$.

5. Proceeding as in the previous problem, you should find that

$$w_x = -x^{-2} w_u, \qquad w_y = y^{-2} w_u - y^{-2} w_v, \qquad w_z = z^{-2} w_v.$$

6. Since both $f_x(x, y) = 4x - 4$ and $f_y(x, y) = 6y$ exist at all points, you need to check the critical points and the points on the boundary. The only critical point is $(1, 0)$, while

$$y^2 = 9 - x^2 \quad \Longrightarrow \quad f(x, y) = 2x^2 - 4x + 3(9 - x^2) = -x^2 - 4x + 27$$

on the boundary. You now need to find the minimum value of this function on $[-3, 3]$.

7. There are two critical points, namely $(0, 0)$ and $(1, 1)$. To check the boundary points, you will need to consider the three sides of the triangle separately. On the horizontal side,

$$y = 0 \quad \Longrightarrow \quad f(x, y) = x^3$$

and we have $0 \leq x \leq 1$, so the maximum value is $f(1, 0) = 1$. On the vertical side,

$$x = 1 \quad \Longrightarrow \quad f(x, y) = y^3 - 3y + 1$$

so you need to find the maximum value of this function on $[0, 2]$. On the hypotenuse,

$$y = 2x \quad \Longrightarrow \quad f(x, y) = x^3 + (2x)^3 - 3x(2x) = 9x^3 - 6x^2$$

so you need to find the maximum value of this function on $[0, 1]$.