MA121 Tutorial Problems #7

1. Letting $\alpha \in \mathbb{R}$ be fixed, find the radius of convergence of the binomial series

$$f(x) = \sum_{n=0}^{\infty} \frac{\alpha(\alpha-1)(\alpha-2)\cdots(\alpha-n+1)}{n!} \cdot x^n$$

2. Compute each of the following sums in terms of known functions:

$$\sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+1}}{n!}, \qquad \sum_{n=1}^{\infty} \frac{(-1)^n x^{4n+1}}{(2n+1)!}, \qquad \sum_{n=0}^{\infty} \frac{(-1)^n x^{6n}}{(2n+2)!}.$$

3. Find the area of the region that lies between the graphs of f(x) = x + 2 and g(x) = x².
4. Find the volume of a sphere of radius r.

5. Let R be the region between the graph of $f(x) = \sin x$ and the x-axis over $[0, \pi]$. Find

- (a) the area of the region R;
- (b) the volume of the solid obtained by rotating R around the x-axis.

- These are all practice problems, not a homework assignment.
- However, part of your next homework assignment will be based on these problems.
- In case you get stuck, some hints are provided on the other page of this sheet.

Hints and comments

1. One always uses the ratio test to check power series for convergence. In this case,

$$L = \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \left| \frac{\alpha - n}{n+1} \cdot x \right|$$

Use the fact that the limit of an absolute value is the absolute value of the limit.

2. In each case, you are supposed to manipulate some of the known Taylor series

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \qquad \sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}, \qquad \cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}.$$

When it comes to the first sum, for instance, one can easily see that

$$\sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+1}}{n!} = x \sum_{n=0}^{\infty} \frac{(-x^4)^n}{n!} = x e^{-x^4}.$$

3. The two graphs intersect when x = -1 and also when x = 2. Using a rough sketch of the graphs, we see that the graph of f lies above the one of g, so the desired area is

Area =
$$\int_{-1}^{2} [f(x) - g(x)] dx = \dots = \frac{9}{2}$$
.

4. Let R be the region between the graph of $f(x) = \sqrt{r^2 - x^2}$ and the x-axis over [-r, r]. Since a sphere of radius r arises by rotating R around the x-axis, its volume is

Volume =
$$\int_{-r}^{r} \pi f(x)^2 dx = \dots = \frac{4\pi r^3}{3}$$
.

5a. Since $f(x) = \sin x$ is both non-negative and continuous on $[0, \pi]$, the desired area is

Area =
$$\int_0^\pi \sin x \, dx = \dots = 2.$$

5b. According to our formula, the desired volume is

Volume =
$$\pi \int_0^{\pi} \sin^2 x \, dx = \pi \int_0^{\pi} \frac{1 - \cos(2x)}{2} \, dx = \dots = \frac{\pi^2}{2}$$