MA121 Tutorial Problems #4

1. Show that $\sin^2 x + \cos^2 x = 1$ for all $x \in \mathbb{R}$.

2. Let f be a non-negative function which is integrable on $[0,1]$ with $f(x) = 0$ for all $x \in \mathbb{Q}$. Show that $\int_0^1 f(x)\,dx = 0$.

3. Suppose f is continuous on $[a,b]$. Show that there exists some $c \in (a,b)$ such that

$$\int_a^b f(t)\,dt = (b-a)\cdot f(c).$$

As a hint, apply the mean value theorem to the function $F(x) = \int_a^x f(t)\,dt$.

4. Compute each of the following integrals:

$$\int \frac{\sin(1/x)}{x^2}\,dx, \qquad \int (x+1)(x+2)^5\,dx, \qquad \int \frac{x}{\sqrt{x+1}}\,dx, \qquad \int xe^x\,dx.$$

5. Compute each of the following integrals:

$$\int \sin^3 x\,dx, \qquad \int \frac{x}{e^x}\,dx, \qquad \int e^{\sqrt{x}}\,dx, \qquad \int \frac{\log x}{x^2}\,dx.$$

- These are all practice problems, not a homework assignment.
- However, part of your next homework assignment will be based on these problems.
- In case you get stuck, some hints are provided on the other page of this sheet.

Hints and comments

1. First of all, show that $f(x) = \sin^2 x + \cos^2 x$ is constant. You will need to know that

$$(\sin x)' = \cos x, \qquad (\cos x)' = -\sin x, \qquad \sin 0 = 0, \qquad \cos 0 = 1.$$

2. Show that $S^-(f, P) = 0$ for all partitions P and recall that $\int_0^1 f(x)\,dx = \sup S^-(f, P)$.

3. According to the mean value theorem, there exists some $c \in (a, b)$ such that

$$\frac{F(b) - F(a)}{b - a} = F'(c).$$

Simplify this equation by noting that $F(a) = 0$ and that $F'(x) = f(x)$ for all x.

4a. Use the substitution $u = 1/x = x^{-1}$, which gives $du = -x^{-2}\,dx$.

4b. Use the substitution $u = x + 2$. Split the resulting integral into two parts.

4c. Use the substitution $u = x + 1$. Split the resulting integral into two parts.

4d. Use either integration by parts or else tabular integration.

5a. Write $\sin^3 x = (1 - \cos^2 x)\sin x$ and then split the given integral into two parts.

5b. You need to integrate xe^{-x}. Use either integration by parts or else tabular integration.

5c. Use the substitution $u = \sqrt{x}$. This gives $x = u^2$ so that $dx = 2u\,du$ and

$$\int e^{\sqrt{x}}\,dx = 2\int ue^u\,du.$$

Next, note that you have already computed the rightmost integral; see question 4.

5d. You need to integrate $x^{-2}\log x$. Use integration by parts, starting with the equation

$$\int x^{-2} \cdot \log x\,dx = \int \left(-x^{-1}\right)' \cdot \log x\,dx.$$